Renormalization Hopf algebra, multiple zeta values and mixed Tate motives

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Goals

1.Review the the schematic diagram

$$\begin{array}{ccccc}
X_\Gamma & \longrightarrow & H^*(\widetilde{X_\Gamma}) & \longrightarrow & \mathcal{MTM} \\
\uparrow & & \downarrow & & \| \\
\Gamma & \longrightarrow & \mathrm{MZV} & \longleftarrow & \mathcal{MTM} \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma \otimes \Gamma & \longrightarrow & \mathrm{MZV} \otimes \mathrm{MZV} & \longleftarrow & \mathcal{MTM} \otimes \mathcal{MTM}
\end{array}$$

2. Compare the coproducts in order to comment on their “compatibility”.

3. (most important!) Convey what motives are all about and what they can do for quantum field theory and multiple zeta values.

Hopf algebra of Feynman graphs (Connes-Kreimer)

Recall that the Connes-Kreimer Hopf algebra is a commutative, noncocommutative graded Hopf algebra with the coproduct defined as

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma//\gamma.$$

Examples: [INSERT EXAMPLES]

Multiple zeta values and iterated integrals

Define an iterated integral as

$$I_\gamma(a_0; a_1, \ldots, a_n; a_{n+1})$$

$$:= (2\pi i)^{-n} \int_{\Delta_{n,\gamma}} \frac{dt_1}{t_1 - a_1} \wedge \ldots \wedge \frac{dt_n}{t_n - a_n}.$$

Here

γ: path from a_0 to a_{n+1} in $\mathbb{C}\ a_1 \cup \ldots \cup a_n$

and the simplex of integration

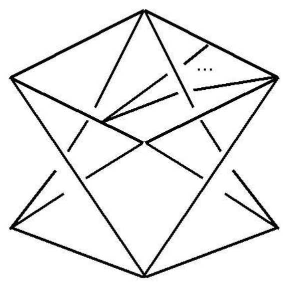
$\Delta_{n,\gamma}$ consists of all n-tuples of points $(t_1, \ldots, t_n)$ on γ.

II's provide integral representations of multiple zeta values:

$$\zeta(m) = \int_{0 \le t_1 \le \ldots t_m \le 1} \frac{dt_1}{1 - t_1} \wedge \frac{dt_2}{t_2} \wedge \ldots \frac{dt_m}{t_m},$$

$$\zeta(m, n) = \sum_{0 < k_1 < k_2} \frac{1}{k_1^m k_2^n} = I(0; 1, 0, \ldots 0, 1, 0, \ldots, 0; 1).$$

Double zetas occur in nature: for # of cycles $= n$,



$\sim \zeta(5, 2n+3)$

It is also well known that MZVs have interesting algebraic properties. For example,

$$I_\gamma(a; z_1, \ldots, z_m; b) \cdot I_\gamma(a; z_{m+1}, \ldots z_{m+n}; b)$$

$$= \sum_{\sigma \in \Sigma_{m,n}} I_\gamma(a; z_{\sigma(1)}, \ldots, z_{\sigma(m+n)}; b)$$

Can describe relations between MZVs in the context of Connes-Kreimer theory:

- Shuffle relations for regularized integrals of symbols(Paycha, Manchon and others)

- Heat kernel methods and Rota-Baxter operators (Hoffman, Guo and others)

Pure Motives

Basics on algebraic cycles

1. X smooth projective schemes over k. $C^i(X)$ abelian gp of algebraic cycles on X of codim i.

2. $\sim$ "adequate" equivalence relation on the cycles of X.

3. $C^i_\sim(X) := C^i(X)/\sim$ andCorr$_\sim(X, Y) := C^i(X \times Y) \otimes \mathbb{Q}$.

4. Corr$^r_\sim(X, Y) := C^{d+r}_\sim(X \times_k Y) \otimes \mathbb{Q}$ where d is the dimension of X.

Define the composition of correspondences:

$$g \bullet f := pr_{XZ}\{(f \times Z) \cap (X \times g)\},$$

where $f \in$ Corr$_\sim(X, Y), g \in$ Corr$_\sim(Y, Z)$.

A correspondence p is a projector if $p \bullet p = p$.

With this we can define the category of <u>pure motives</u>:

- objects: (X, p, m) where p is a projector and $m \in \mathbb{Z}$ "twist".

- morphisms: Hom(M, N) := $q \bullet C^{n-m}_{\sim}(X, Y) \bullet p$. Here $M = (X, p, m)$ and $Y = (Y, q, n)$ are pure motives.

The idea is that the following diagram should hold: [INSERT DIAGRAM]

Neutral Tannakian category $\mathcal{T}$

Rigid $\mathbb{Q}$-linear, abelian tensor category with internal homs. with fiber functor $\mathcal{T} \longrightarrow \text{Vect}_k$.

Nice result due to Jannsen (assuming the "standard conjectures" $\sim_{\text{num}} = \sim_{\text{hom}}$):

The category of pure motives (with respect to $\sim_{num}$) is neutral Tannakian.

Examples of pure motives:

1. $(\text{Spec } k, \text{id}, 0)$: motive of a point.

2. $(\text{Spec } k, \text{id}, -1)$: Lefschetz motive.

3. $(\text{Spec } k, \text{id}, 1)$: Tate motive.

Mixed Hodge structures (Deligne)

Pure Hodge structure of wt. m on a f.d. vector space V is a decreasing filtration

$$\ldots \subset F^{p+1}V_{\mathbb{C}} \subset F^p V_{\mathbb{C}} \subset \ldots$$

($V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$) satisfying (Hodge decomposition)

$$V_{\mathbb{C}} = \bigoplus_{p+q=m} V^{p,q},$$

where

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^p V_{\mathbb{C}}}.$$

An A-mixed Hodge structure ($A \subset \mathbb{R}$) consists of

- An A-module of finite type V_A,
- Incr. filt. ("wt. filtration")

  $$\ldots \subset W_n \subset W_{n+1} \subset \ldots$$

  of $A \otimes \mathbb{Q}$- module $V_A \otimes \mathbb{Q}$,

- Decr. filt. ("Hodge filtration")
$$\ldots \subset F^{p+1}V_{\mathbb{C}} \subset F^p V_{\mathbb{C}} \subset \ldots$$
where $V_{\mathbb{C}} = V_A \otimes \mathbb{C}$.

Define the graded weight j factor $\mathrm{gr}_j^W(V_A) := (W_j/W_{j-1}) \otimes \mathbb{C}$ and require the data to satisfy

F and $\bar{F}$ on $V_{\mathbb{C}}$ induce an pure Hodge structure of wt. j on $\mathrm{gr}_j^W(V_A)$.

Morphisms: $f : V_A \longrightarrow V'_A$ are A-module homs that are compatible with the filtrations W and F.

$\rightsquigarrow$ category of A-mixed Hodge structures.

Important fact (due to Deligne):

This category is abelian with kernels and cokernels with induced filtrations.

Mixed motives: rough idea (Beilinson, Voevodsky, . . .)

Want an abelian category of mixed motives (i.e., motives of smooth schemes over any arbitary field.)

No candidate exists!

However, there is a *triangulated* derived category of mixed motives (Voevodsky and others).
Properties of the conjectural abelian category $\mathcal{MM}$ (Beilinson):

- Subcategory of semisimple objects of $\mathcal{MM}$ is the category of homological pure motives $\mathcal{M}_{\sim_{\text{hom}}}$.

- $M \in \text{Obj } \mathcal{MM}$ has a wt. filtration (via the functor $\mathcal{MM} \longrightarrow \mathbb{Q} - \text{MHS}$) with graded factors in $\mathcal{M}_{\sim_{\text{hom}}}$.

- $\text{Ext}^i_{\mathcal{MM}}(\cdot,\cdot) = 0$ for $i > 1$.

There is a category that <u>does</u> exists that would be a subcategory of $\mathcal{MM}$

<u>Mixed Tate motives $\mathcal{MTM}$ over a number field F</u>

- Start with the (pure) Tate motive $\mathbb{Q}(1)$ and write $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$.

- Assume that $\mathbb{Q}(a)$ and $\mathbb{Q}(b)$ are nonisom. for $a \neq b$ and that any simple obj. of $\mathcal{MTM}$ is isom to some $\mathbb{Q}(\cdot)$.

Also consider the extensions $\text{Ext}^i_{\mathcal{MM}}(\mathbb{Q}(0), \mathbb{Q}(n))$ and assume that they vanish for $i > 1$.

Borel:

$$\text{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(F) \otimes \mathbb{Q}.$$

Here $K_\bullet(\cdot)$ is algebraic K-theory (in the sense of Quillen): $K_\bullet(F) \otimes \mathbb{Q} = H_\bullet(GL(n,F))/\text{indecompos.}$

Fact:

The category of mixed Tate motives over a number field is a Tannakian category with objects $\mathbb{Q}(n)$ with $n \in \mathbb{Z}$ ("simple objects") and the extensions described above. This category is generated by the simple objects.

Furthermore, there is a "Hodge realization" functor

$$\mathcal{MTM} \longrightarrow \mathbb{Q} - \text{MHS}$$

which takes a simple object to a 1-dim $\mathbb{Q}$-vector space with MHS.

Deligne-Goncharov: *This functor for a #-field is exact and faithful.* With this, we have a concrete description of the functor for extensions:

It takes a short exact sequence in $\mathcal{MTM}$ to a short exact sequence of vector spaces with MHS:

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow E_n \longrightarrow \mathbb{Q}(n) \longrightarrow 0.$$

Periods (Kontsevich-Zagier)

Input data (X, D, ω, γ):

- X: smooth d-dim'l algebraic variety over $\mathbb{Q}$.
- $D \subset X$: divisor on X with normal crossings.
- $\omega \in \Omega^d(X)$: alg. diff. form on X, $d^2\omega = 0$.
- $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$: rel. homology class of sing. chain on $X(\mathbb{C})$ with boundary in $D(\mathbb{C})$.

The period integral is defined as

$$\int_\gamma \omega \in \mathbb{C}.$$

Mixed Tate motives and MZVs

General conjecture about $\mathcal{MTM}$ due to Goncharov:

Let $M \in \text{Obj } \mathcal{MTM}$. Then the period of M is a multizeta value (more precisely, in $\mathbf{MZV}[\frac{1}{2\pi i}]$).

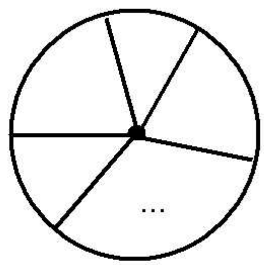
Expectation: associate to a Feynman graph Γ, a motive $[X_\Gamma]$.

If $[X_\Gamma]$ mixed Tate, then see if the period (obtained by Feynman rules) is MZV or not.

Remark: In general $\forall\Gamma$, $[X_\Gamma]$ is NOT mixed Tate (by the work of Belkale-Brosnan).

Mixed Tate motives in nature (Bloch-Esnault-Kreimer)

Consider the Feynman graph ("wheels with n-spokes") from scalar field theory with # of cycles = n



$\sim \zeta(2n-3)$

By considering the Kirchoff polynomial $\Psi = \sum_T \prod_{e \notin T} A_e$ of this graph, one can get a variety X_n.

Theorem:

$$H^{2n-1}(\mathbb{P}^{2n-1} \backslash X_n) \simeq \mathbb{Q}(2n-3)$$

$$\text{equiv. } H^{2n-1}(X_n, \mathbb{Q})_{\text{prim}} \simeq \mathbb{Q}(-2)$$

where

$$H^{2n-1}(X_n, \mathbb{Q})_{\text{prim}}$$

$$:= \text{coker}(H^{2n-1}(X_n, \mathbb{Q}) \longrightarrow H^*(X_n, \mathbb{Q}))$$

Hopf algebra of mixed Tate motives (Goncharov)

$\mathcal{M}$: mixed Tate category over char 0 field K (same as above, except simple objects are from any Tannakian category) and $n \geq 0$.

$M \in \mathrm{Obj}\ \mathcal{M}$ is $\underline{n\text{-framed}}$ (M, v_0, f_n) if there are nonzero morphisms ("frames")

- $v_0 : K(0) \longrightarrow \mathrm{gr}_0^W M$,
- $f_n : \mathrm{gr}_{-2n}^W M \longrightarrow K(n)$.

We say $M_1 \sim M_2$ if there are nonzero morphisms $M_1 \longrightarrow M_2$ that respect the frames.

$\mathcal{A}_n(\mathcal{M})$: set of equiv. classes.

$[M, v_0, f_n] + [M', v_0', f_n'] := [M \oplus M', (v_0, v_0'), f_n + f_n']$.

$\rightsquigarrow \mathcal{A}_n$ is an abelian group.

Furthermore, $\mathcal{A}_n$ is a graded $\mathbb{Q}$-Hopf algebra. Define

$$\Delta = \bigoplus_{0\le p\le n} \Delta_{p,n-p} : \mathcal{A}_n(\mathcal{M})$$

$$\longrightarrow \bigoplus_{0\le p\le n} \mathcal{A}_p(\mathcal{M}) \otimes \mathcal{A}_{n-p}(\mathcal{M})$$

in the following way:

Choose the b_i of $\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(p), \mathrm{gr}^W_{-2p})$ and the dual basis $b_i^{'}$ of $\mathrm{Hom}_{\mathcal{M}}(\mathrm{gr}^W_{-2p}, \mathbb{Q}(p))$ for $1 \le i \le m$. The coproduct explicitly is

$$\Delta_{p,n-p}[M, v_0, f_n] := \sum_{i=1}^{m} [M, v_0, b_i^{'}] \otimes [M, b_i, f_n](-p).$$

The Hopf algebra commutative but not necc. cocommutative.

Hopf algebra of motivic iterated integrals: brief comments (Goncharov)

Instead of dealing with II's, work with motivic iterated integrals.

Use the isom.

$$\mathcal{A}_\bullet(F) \cong T(\oplus_{n\geq 1} K_{2n-1}(F)\otimes\mathbb{Q}),$$

the tensor algebra of the graded $\mathbb{Q}$-vector space. Goncharov shows that for an embedding $\sigma : F \hookrightarrow \mathbb{C}$, there is a filtered graded algebra $P^\sigma(F)$ such that

$$p_\sigma : \mathcal{A}_\bullet(F) \longrightarrow P^\sigma(F)$$

is a surjection of algebras. Using this, he defines the motivic iterated integrals

$$I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{A}_n(F)$$

such that

$$I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1})$$

$$= \bar{I}(\sigma(a_0); \sigma(a_1), \ldots, \sigma(a_n); \sigma(a_{n+1}))$$

Here

$$\bar{I}(\sigma(a_0); \sigma(a_1), \ldots, \sigma(a_n); \sigma(a_{n+1}))$$

is an element in $P^\sigma(F)$.

Example:

$$I(a,b,c) = (2\pi i)^{-1} \int_a^c \frac{dt}{t-b} = (2\pi i)^{-1} \log \frac{c-b}{a-b}$$

After regularizing in ϵ and taking the free term of the poly in $\log \epsilon$, one can show that

$$I(a,b,c) = (2\pi i)^{-1} \log \tilde{r}(a,b,c)$$

The coproduct on the motivic iterated integrals is given by

$$\Delta I^{\mathcal{M}}(a_0; a_i, a_2, \ldots, a_n; a_{n+1}) =$$

$$\sum_{0 \le i_0 < i_1 < \ldots < i_{k+1} = n+1} I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1})$$

$$\otimes \prod_{p=0}^{k} I^{\mathcal{M}}(a_{i_p}; a_{i_{p+1}}, \ldots, a_{i_{p+1}-1}; a_{i_p+1}).$$

Basic references

1. Bloch-Esnault-Kreimer: "On motives associated to graph polynomials"

2. Goncharov: "Galois symmetries of fundamental groupoids and noncommutative geometry"

3. Numerous papers by Broadhurst-Kreimer on higher loop calculations

4. Lectures of Bloch, Esnault, Goncharov and Kreimer at the IHES workshop in June 2006