Polylogarithms

The classical, the modern and the avant-garde

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Most of what I am going to tell you, in some cases, go back to Euler. In any case, I learned this material though conversations with, and by reading the writings of, Don Zagier, one of my Bonn teachers.

What follows are the ideas of Zagier and his collaborators and in more than many places, I have adopted their work verbatim. The last section, though, is one in which I have tried to contribute something to.

The sole purpose of this "digest" is to inform an audience of this work who might have not known of it otherwise and hopefully excite some student's imagination!

1. Polylogarithms: definitions and some identities

Consider the classic expansion for $z \in \mathbb{C}$:

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \text{ with } |z| < 1.$$

The polylogarithm is defined through the same way through analogy:

$$\boxed{\text{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \text{ for } |z| < 1 \text{ and } m \in \mathbb{N}}$$

Look at the special case of $m = 2$.

$$\boxed{\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ for } |z| < 1.}$$

This is called the dilogarithm.

There are some facts of life with these functions:

1. Domain of definition can be extended to $\mathbb{C} \setminus [1, \infty)$ by induction on
$$\frac{d}{dz}\mathrm{Li}_m(z) = \frac{1}{z}\mathrm{Li}_{m-1}(z) \text{ for } m \geq 2.$$

2. $\mathrm{Li}_2(z)$ has an integral representation:
$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u} \text{ for } z \in \mathbb{C} \setminus [1, \infty).$$

We are interested in the special values of $\mathrm{Li}_2(z)$, that is, values of z which the function in integer or a rational number or for that matter an irrational number.

The special values of analytic functions is an old topic!

For example, the Γ-function

$$\Gamma(s) = \int_0^s t^{s-1} e^{-t} dt$$

has, as natural numbers $n \in \mathbb{N}$

$$\begin{aligned} \Gamma(n) &= (n-1)!, \\ \Gamma\left(n + \frac{1}{2}\right) &= \frac{(2n)!!}{4^n n!}\sqrt{n}. \end{aligned}$$

The Riemann-Euler(RE) zeta function, for a complex number s, is defined as

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.}$$

The RE zeta functions have some very special values, taking the argument as integers. As examples:

$\zeta(2) = \frac{\pi^2}{6}$	$\zeta(4) = \frac{\pi^4}{90}$	$\zeta(6) = \frac{\pi^6}{945}$	...
$\zeta(0) = -\frac{1}{2}$	$\zeta(-2) = 0$	$\zeta(-4) = 0$	...
$\zeta(-1) = -\frac{1}{12}$	$\zeta(-3) = \frac{1}{120}$	$\zeta(-5) = -\frac{1}{252}$	...

We can have infinite number of such values for the RE zeta function, in contrast with . . .

the fact that there are only eight known values for which both z and $\mathrm{Li}_2(z)$ can be written in closed form:

$$
\begin{aligned}
\mathrm{Li}_2(0) &= 0 \\
\mathrm{Li}_2(1) &= \frac{\pi^2}{6} \\
\mathrm{Li}_2(-1) &= -\frac{\pi^2}{12} \\
\mathrm{Li}_2\Big(\frac{1}{2}\Big) &= \frac{\pi^2}{12} - \frac{1}{2}\log^2(2) \\
\mathrm{Li}_2\Big(\frac{3-\sqrt{5}}{2}\Big) &= \frac{\pi^2}{15} - \log^2\Big(\frac{1+\sqrt{5}}{2}\Big) \\
\mathrm{Li}_2\Big(-\frac{1+\sqrt{5}}{2}\Big) &= \frac{\pi^2}{10} - \log^2\Big(\frac{1+\sqrt{5}}{2}\Big) \\
\mathrm{Li}_2\Big(\frac{1-\sqrt{5}}{2}\Big) &= -\frac{\pi^2}{15} + \frac{1}{2}\log^2\Big(\frac{1+\sqrt{5}}{2}\Big) \\
\mathrm{Li}_2\Big(-\frac{1-\sqrt{5}}{2}\Big) &= -\frac{\pi^2}{10} + \frac{1}{2}\log^2\Big(\frac{1+\sqrt{5}}{2}\Big)
\end{aligned}
$$

The dilogarithm satisfy some not-so-interesting functional equations:

$$\begin{aligned}\mathrm{Li}_2\Big(\frac{1}{z}\Big) &= -\mathrm{Li}_2(z) - \Big(\frac{\pi^2}{6} + \frac{1}{2}\log^2(-z)\Big),\\ \mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z) - \Big(\frac{\pi^2}{6} + \frac{1}{2}\log(1-z)\log(z)\Big).\end{aligned}$$

We can modify the functional equations above to more symmetric forms by changing the definition of the dilogarithm (which we'd do in a short while!) a little, and also bring it into the circle of contemporary ideas.

Most crucially the dilogarithm satisfies the 5-term relationship:

$$\begin{aligned} &\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\Big(\frac{1-x}{1-xy}\Big) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\Big(\frac{1-y}{1-xy}\Big)\\ = \ & \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y)\\ + \ & \log\Big(\frac{1-x}{1-xy}\Big)\log\Big(\frac{1-y}{1-xy}\Big).\end{aligned}$$

2. Variants of the dilogarithm

Bloch–Wigner dilogarithm

$$\boxed{D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z)}$$

Roger dilogarithm

For $x \in (0,1)$ define

$$\boxed{L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x)}$$

and extend to all of $\mathbb{R}$ by defining $L(0) = 0$ and $L(1) = \frac{\pi^2}{6}$ by setting

$$L(x) = \begin{cases} 2L(1) - L\left(\frac{1}{x}\right) & \text{if } x > 1 \\ -L\left(\frac{x}{x-1}\right) & \text{if } x < 0. \end{cases}$$

Define a map $\mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{R}/\mathbb{Z} \mod \frac{\pi^2}{2}$ given by:

$$\boxed{\bar{L}(x) := L(x) \mod \frac{\pi^2}{2}}$$

This newly constructed map has very nice features:

Additive functional equation $\bar{L}(x) + \bar{L}(1-x) = \bar{L}(1)$

Mutliplicative functional equation $\bar{L}(x) + \bar{L}\left(\frac{1}{x}\right) = -\bar{L}(1)$

They also satisfy the 5-term identity

$$\bar{L}(x) + \bar{L}(y) + \bar{L}(1-xy) + \bar{L}\left(\frac{1-x}{1-xy}\right) + \bar{L}\left(\frac{1-x}{1-xy}\right) = 0.$$

3. Volumes in hyperbolic space and K-theory

Recall the Bloch–Wigner dilogarithm:

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z).$$

They satisfy the 5-term relationship in the same way $\bar{L}$ does.

Now define

$$\widetilde{D}(z_0, z_1, z_2, z_3) := D\left(\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}\right).$$

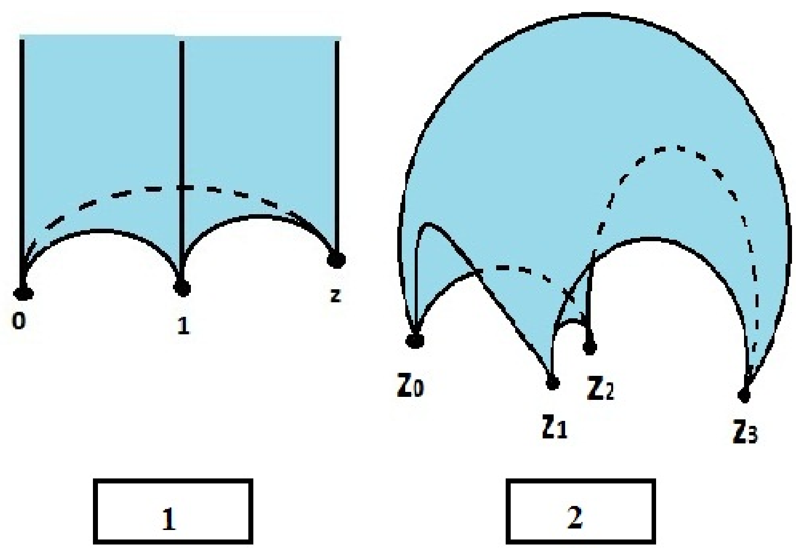
Then the 5-term identity becomes

$$\boxed{\sum_{i=0}^{4}(-1)^i \widetilde{D}(z_0, \ldots, \widehat{z}_i, \ldots, z_4) = 0}$$

where $z_0, \ldots, z_4 \in \mathbb{P}^1_{\mathbb{C}}$.

These two functions D and $\widetilde{D}$ gives us volumes of ideal tetrahedra:

$$\begin{aligned}\text{Vol of } \boxed{1} &= D(z) \text{ with coordinates } 0, 1, z, \infty,\\ \text{Vol of } \boxed{2} &= \widetilde{D}(z_0, z_1, z_2, z_3) \text{ with coordinates } z_0, z_1, z_2, z_3.\end{aligned}$$



Fact. We can decompose any hyperbolic 3-manifold as sums of ideal tetrahedra. Denote this set of tetrahedra as $\{\Delta_1, \ldots, \Delta_n\}$ and the Δ_v (with $1 \leq v \leq n$) has vertices $0, 1, \infty, z_v$. Then for any hyperbolic 3-manifold M,

$$\boxed{\text{Vol}(M) = \sum_{v=1}^{n} \text{Vol}(\Delta_v) = \sum_{v=1}^{n} D(z_v)}$$

The parameters z_v triangulating a complete hyperbolic 3-manifold satisfy

$$\sum_{v=1}^{n} \underbrace{z_v \wedge (1 - z_v)}_{\in \bigwedge^2 \mathbb{C}^*} = 0$$

The exterior algebra has the explicit description

$$\bigwedge^2 \mathbb{C}^* = \{\text{lin. comb. of } x \wedge y \in \mathbb{C}^* | x \wedge x = 0, (x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y\}.$$

Corollary. The set $\{D(z_v)\}$ is countable since every volume of hyperbolic 3-manifolds can be expressed in terms of $D(z)$.

We can create an abstract version of the above construction–
Let $\widetilde{\mathcal{B}}_{\mathbb{C}}$ be an abelian group of sums $[z_1] + \ldots + [z_n]$ with $z_1, \ldots, z_n \in \mathbb{C}^* \setminus \{1\}$ satisfying

$$\sum_i z_i \wedge (1 - z_i) = 0$$

$\widetilde{\mathcal{B}}_{\mathbb{C}}$ contains the elements

$$\mathcal{I} = \left\{ [x] + \left[\frac{1}{x}\right], [x] + [1-x], [x] + [y] + \left[\frac{1-x}{1-xy}\right], [1-xy] + \left[\frac{1-y}{1-xy}\right] \right\}.$$

We obtain the Bloch group by taking the quotient

$$\boxed{\mathcal{B}_{\mathbb{C}} = \widetilde{\mathcal{B}}_{\mathbb{C}}/I}$$

For the experts: there is a natural embedding $\mathcal{B}_{\mathbb{C}} \hookrightarrow K_2(\mathbb{C})$.

The Bloch group is of immense importance all over mathematics and physics. As a (sophisticated example), look at the map

$$(2\pi i)^2 L : \mathcal{B}_{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}.$$

This map takes values in $\mathbb{Q}/\mathbb{Z}$ on torsion elements and are called conformal dimensions in string theory.

4. Multiple zeta values

Multiple polylogarithms

Take n_i and k_j to be integers for $1 < i, j < m$.

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \sum_{0<n_1<\ldots<n_m} \frac{x_1^{n_1}\cdots x_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}$$

A multiple zeta value (MZV) is a solution to the above equation where $x_1 = x_2 = \cdots = x_m = 1$ with values between $\bar{\mathbb{Q}}$ and $\mathbb{C}$. The corresponding multiple zeta function is therefore defined as

$$\zeta(k_1,\ldots,k_n) = \sum_{0<n_1<\ldots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

This is an obvious generalization of the Riemann–Euler zeta function.

This area of number theory is specially filled with very interesting identities such as

$$\begin{aligned} \zeta(2,1) &= \zeta(3), \\ \zeta(a,b)+\zeta(b,a) &= \zeta(a)\zeta(b)-\zeta(a,b) \text{ for } a,b>1. \end{aligned}$$

There are two main invariants to the study of MZVs:

$$\boxed{[\text{Weight}]\ \mathbf{k} := k_1 + \ldots + k_m}$$

and

$$\boxed{[\text{Depth}]\ \mathbf{n} := n_1 + \ldots + n_m}$$

A very easy example

Weight	Function
1	$\mathrm{Li}_1(x) = -\log(1-x)$
2	$\mathrm{Li}_2(x)$ and $\mathrm{Li}_{1,1}(x,y)$

There are several very interesting identities between polylogarithms; for example- for $x, y \in \mathbb{C}$, $|xy| < 1$ and $|y| < 1$ we have:

$$\begin{aligned}
\mathrm{Li}_{1,1}(x,y) &= \mathrm{Li}_2\Big(\frac{xy-y}{1-y}\Big) - \mathrm{Li}_2(xy) - \mathrm{Li}_2\Big(\frac{-y}{1-y}\Big),\\
\mathrm{Li}_{1,1}(x,y) + \mathrm{Li}_{1,1}(y,x) + \mathrm{Li}_2(x,y) &= \mathrm{Li}_1(x)\mathrm{Li}_1(y),\\
\mathrm{Li}_{1,1}(x,y) &= \mathrm{Li}_1(x)\mathrm{Li}_1(y) + \mathrm{Li}_2\Big(\frac{-x}{1-x}\Big) - \mathrm{Li}_2\Big(\frac{xy-x}{1-x}\Big).
\end{aligned}$$

Very difficult general open question

> For a fixed weight, find all relationship between $\zeta(k_1, \ldots, k_m)$ and lower weight values.

This is related to Zagier's conjecture: Can one compute the dimension of the $\mathbb{Q}$-vector space spanned by MZVs in terms of their weight and depth?

One very new line of attack on this problem has Feynman integrals of perturbative quantum field theory through the work of Bloch, Broadhurst, Brown and Kreimer.

5. Feynman diagrams and multiple zeta values

Massless scalar quantum field theory in dimension D given by the Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{1}{k!}\phi^k.$$

We will vary k and D to generate interesting results.

The parametric form of a Feynman integral in D dimension and associated to a Feynman graph Γ with n edges and l loops and with incoming external momenta p is, up to some constant, given by

$$\boxed{U(\Gamma) = \frac{\Gamma(n - \frac{Dl}{2})}{(4\pi)^{Dl/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(p,t)^{n-Dl/2}}}$$

Dealing with the simpler case of $p = 0$, the main object of focus in the integrand is the Kirchhoff polynomial

$$\boxed{\Psi_\Gamma(t) = \sum_{T \subset \Gamma} \prod_{e \notin T} t_e}$$

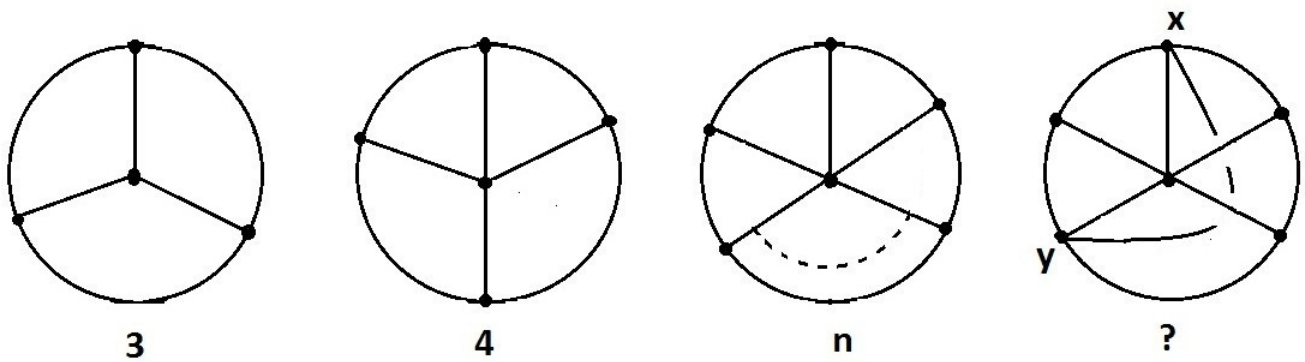
This homogeneous polynomial has very interesting combinatorics and is the main "work horse" in evaluating the Feynman integral.

These integrals in general always diverge! Physics machinery called renormalization manages to extract some finite number from them!

It has been a striking observation from the mid-nineties of Broadhurst and Kreimer that a large class of Feynman integrals, after renormalization, systematically evaluate to zeta and multiple zeta values!

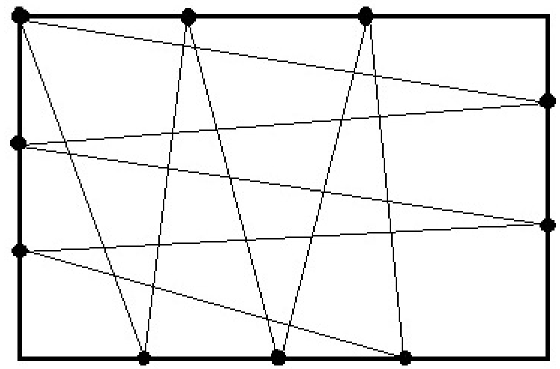
In the following panel of Feynman graphs,

Graph 1 with 3 spokes evaluate to	$6\zeta(3)$
Graph 2 with 4 spokes evaluate to	$20\zeta(5)$
Graph 3 with n spokes evaluate to	$c_n\zeta(2n-3)$



On the other hand, combinatorially sensitive– adding one more edge in the last graph in a different way completely throws off the numerical value!

Multiple zeta values (as claimed!) also show up (of course, after tremendous effort using computers and combinatorics!) For example, Schnetz show that



evaluates to

$$\frac{149}{2^{13}}\zeta(3) - \frac{7}{2^{11}}\zeta(5,3,5) + \frac{5}{2^{12}}(\zeta(3,7,3) - \zeta(3)\zeta(7,3))$$
$$+ \frac{47}{2^{11}}\zeta(5)\zeta(5,3) - \frac{35}{2^{11}}\zeta(3)^2\zeta(7) + \frac{25}{2^9}\zeta(3)\zeta(5)^2$$
$$+ \frac{5}{2^9}\zeta(5)\zeta(8).$$

A natural question is how does one systematically understand the relationship between graphs and the corresponding integrals.

The most active line of attack is to use the theory of motives in algebraic geometry:

$$\boxed{\Gamma \rightsquigarrow \Psi_\Gamma \rightsquigarrow X_\Gamma := \{\Psi_\Gamma = 0\} \rightsquigarrow H^*(X_\Gamma)}$$

with the hope that the period of the motive $H^*(X_\Gamma)$ will be given by the renormalized value of the Feynman integral.

However, it is still an open question whether one can directly "see" the value from the combinatorics of the graph itself. Some progress has been made in this direction:

1. the $\mathbb{Q}[\frac{1}{2\pi i}]$-vector space of MZVs and Feynman graphs have many formal similarities (Paycha).

2. Brown and Yeats and Brown, Schnetz and Yeats have investigated the "weight-drop phenomena" of MZVs directly at the level of graphs.

3. C. Bergbauer and I have looked at the combinatorics of Ψ_Γ when we insert one graph into another following the Connes-Kreimer Lie algebra of graphs.