Periods, geometry and arithmetic in quantum fields and strings

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Two topics

- Feynman integrals and motives

- Landscape of string vacua

Underlying unifying theme: *periods*.

In this talk, I focus mostly on the first.

Main results

- large classes of Feynman integrals can be viewed as periods of supermanifolds (with *Marcolli*)

- an explicit formula for graph polynomials under insertion (with *Bergbauer*)

- characteristic classes of some graph hypersurfaces

- NP completeness in deciding gauge groups based on rank

- (speculative!) periods, heights, and string compactifications

I

Feynman graphs and motives

Feynman graphs and motives

- *Broadhurst-Kreimer* 90's: large classes of Feynman graphs evaluate to MZV.

- *Deligne-Goncharov*: MZV's are periods of mixed Tate motives. Are there such objects associated to Feynman graphs?

- *Bloch-Esnault-Kreimer*: Yes for WS_n by naturally associated varieties to graphs as zeros of certain polynomials associated to graphs.

- Earlier work due to *Belkale-Brosnan*: motives associated to Feynman graphs generate the (Grothendieck) ring of varieties over $\operatorname{Spec}\mathbb{Z}$ (disproof of a conjecture of *Kontsevich*).

Parametric form of a log-divergent Feynman integral associated to a graph with n edges and in D dimensions:

$$U(\Gamma) = \int_{\Sigma_n} \frac{\Omega}{\Psi_\Gamma^{D/2}}$$

where

$$\Omega = \sum_{i=0}^{n} (-1)^{i+1} t_i \, dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n$$

is the volume form in $\mathbb{P}^{n-1}(\mathbb{R})$, and

$$\Psi_\Gamma(t) := \sum_T \prod_{e \notin T} t_e.$$

(T: a spanning tree of Γ)

Definition

$$X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{P}^{n-1}$$

Basic object of study:

$$(\mathbb{P}^{n-1} \setminus X_\Gamma, \Delta)$$

where $\Delta = \{ \prod_{i=1}^n t_i = 0 \} \supset \partial \Sigma_n$

Properties of X_Γ

- Typically singular with singular locus of small codimension
- Integral diverges whenever $X_\Gamma \cap \Delta \neq \emptyset$ — blow-ups needed!

We will focus on certain problems related to X_Γ and Ψ_Γ.

A main result needed:

$$\Psi_\Gamma(t) = \det M_\Gamma(t),$$

where $M_\Gamma(t)$ is constructed as follows.

Let

- $n = \#E(\Gamma)$, $\ell = b_1(\Gamma)$ (# of loops), $\{l_1, \ldots, l_\ell\}$ basis of $H_1(\Gamma, \mathbb{Z})$

- $$\eta_{ik} = \begin{cases} +1, & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation} \\ -1, & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$(M_\Gamma)_{kr}(t) := \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}$$

for $t = (t_0, \ldots, t_{n-1}) \in \Sigma_n$, $t_n = 1 - \sum_{i=0}^{n-1} t_i$.

- $p_i \in \mathbb{R}^D$: real variables associated to edges of Γ

- $s_k \in \mathbb{R}^D$: real variables associated to loops of Γ

- $q_i(p) := p_i^2 - m^2$: inverse propagator

Upon change of variables $p_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} s_k$ with the constraint $\sum_{i=0}^{n} t_i u_i \eta_{ik} = 0$, plus some manipulations, we have (for $n = D\ell/2$)

$$\int \frac{d^D s_1 \cdots d^D s_\ell}{q_0 \cdots q_n} = C_{\ell,n} \int_{\Sigma_n} \frac{dt_0 \cdots dt_{n-1}}{\det M_\Gamma(t)^{D/2}},$$

where

$$C_{\ell,n} = \int \frac{d^D x_1 \cdots d^D x_\ell}{(1 - \sum_k x_k^2)^n}.$$

Generalization of log-divergent integrals

Theorem — Suppose given a graph with n edges of which f fermionic and $b = n - f$ bosonic. Assume there exists a choice of basis of $H_1(\Gamma, \mathbb{Z})$ satisfying

$$n - \frac{f}{2} + \frac{D}{2}(\ell_f - \ell_b) = 0.$$

Then the following identity holds:

$$\int \frac{\not{q}_1 \cdots \not{q}_f}{q_1 \cdots q_n} d^D s_1 \cdots d^D s_{\ell_b} d^D \sigma_1 \cdots d^D \sigma_{\ell_f} = \int_{\Sigma_n} \frac{\Lambda(t)}{\mathrm{Ber}\, \mathcal{M}(t)^{D/2}} dt_1 \cdots dt_n.$$

Here:

- $q(p) = p^2 - m^2, \quad \not{q}(p) = i(\not{p} + m), \quad \not{p} = p^\mu \gamma_\mu$

- $\Lambda(t)$: uninteresting term depending on t

- $\mathcal{M}(t) = \begin{pmatrix} M_b(t) & \frac{1}{2} M_{fb}(t) \\ \frac{1}{2} M_{bf}(t) & M_f(t) \end{pmatrix}, \quad \mathrm{Ber}\, \mathcal{M} = \frac{\det(M_b - \frac{1}{4} M_{fb} M_f^{-1} M_{bf})}{\det M_f}$

Therefore, in case of theories with bosonic and fermionic legs, the analogue of the log-divergent case is

$$\int_{\Sigma_n} \frac{\Lambda(t)}{\mathrm{Ber}\,\mathcal{M}(t)^{D/2}}\, dt_1 \cdots dt_n$$

for

$$\int_{\Sigma_n} \frac{dt_1 \cdots dt_n}{\det M_\Gamma(t)^{D/2}}.$$

Graph supermanifolds

Divergence when Σ_n intersects with the subvar. of $\mathbb{P}^{n-1}$ defined by

$$\frac{\mathrm{Ber}\,\mathcal{M}(t)^{D/2}}{\Lambda(t)} = 0 \qquad (*)$$

Lemma — The zeros of $(*)$ define a divisor in $\mathbb{P}^{n-1|2f}$ of dim. $(n-2|2f)$. The support of this divisor is the same as that of the principal divisor defined by $\mathrm{Ber}\,\mathcal{M}(t)$.

Def. — Γ: graph with bosonic and fermionic edges; B: basis for $H_1(\Gamma, \mathbb{Z})$. Define

$$\mathcal{X}_{(\Gamma,B)} \subset \mathbb{P}^{n-1|2f}$$

to be the locus of zeros and poles of $\mathrm{Ber}\,\mathcal{M}(t)$.

$\mathcal{X}_{(\Gamma,B)}$ is called the *graph supermanifold*.

Grothendieck ring, motives and supermanifolds

Def. — Let $\mathcal{SV}_{\mathbb{C}}$ be the category of complex supermanifolds. Let $K_0(\mathcal{SV}_{\mathbb{C}})$ denote the free abelian group generated by isom. classes of objects $\mathcal{X} \in \mathcal{SV}_{\mathbb{C}}$ subject to the following:

Let $F\colon \mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed embedding of supermanifolds. Then

$$[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} \setminus \mathcal{Y}],$$

where $\mathcal{X} \setminus \mathcal{Y}$ is the supermanifold

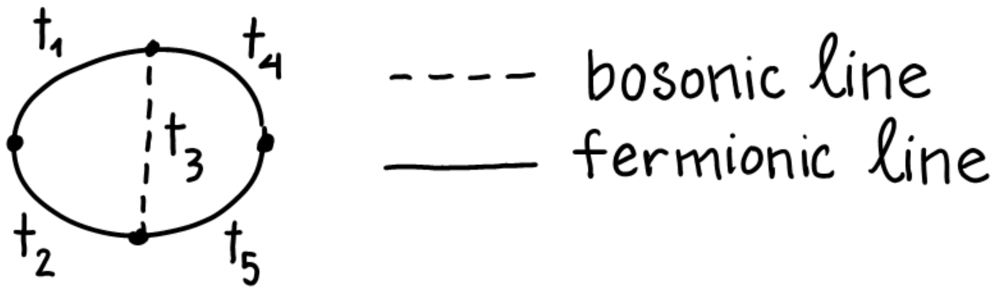
$$\mathcal{X} \setminus \mathcal{Y} = (X \setminus Y, \mathcal{A}_{X|X\setminus Y})$$

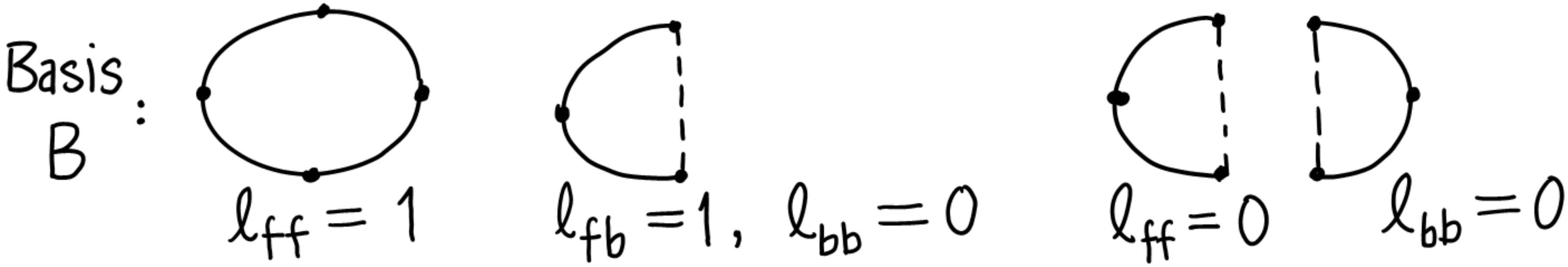
($\mathcal{A}$ is a sheaf of supercommutative rings on X)

Prop. — $K_0(\mathcal{SV}_{\mathbb{C}}) = K_0(\mathcal{V}_{\mathbb{C}})[T]$, $T = [\mathbb{A}^{0|1}]$ class of the affine superspace of dim. $(0,1)$. Also, $K_0(\mathcal{SV}_{\mathbb{C}})/I \cong \mathbb{Z}[SSB]$, where I is the ideal gen. by $[\mathbb{A}^{0|1}]$, $[\mathbb{A}^{1|0}]$.

(There are two different kinds of Lefschetz motives: $\mathbb{L}_f = [\mathbb{A}^{0|1}]$, $\mathbb{L}_b = [\mathbb{A}^{1|0}]$.)

Example





Here

$$M_b(t) = t_1 + t_2 + t_3, \quad M_{bf}(t) = (t_1 + t_2, t_1 + t_2 + t_3), \quad M_f(t) = \begin{pmatrix} 0 & t_1 + t_2 \\ -(t_1 + t_2) & 0 \end{pmatrix}$$

Therefore

$$M_{bf}(t) M_f(t)^{-1} M_{fb}(t) = -(t_1 + t_2 + t_3) + t_1 + t_2 + t_3 \equiv 0$$

and

$$\text{Ber}\, \mathcal{M}(t) = \frac{\det M_b(t)}{\det M_f(t)} = \frac{t_1 + t_2 + t_3}{(t_1 + t_2)^2}.$$

So, $\mathcal{X}_{\Gamma,B} \subset \mathbb{P}^{5|8}$ is the union of $t_1 + t_2 + t_3 = 0$ and $t_1 + t_2 = 0$ in $\mathbb{P}^5$ with restriction of the sheaf from $\mathbb{P}^{5|8}$.

Universality

Prop. — Let $\mathcal{R}$ be the subring of the Grothendieck ring $K_0(\mathcal{SV}_{\mathbb{C}})$ spanned by $[\mathcal{X}_{(\Gamma,B)}]$ for $\mathcal{X}_{(\Gamma,B)}$ given by the zeros and poles of the Berezinian $\mathrm{Ber}\,\mathcal{M}(t)$ with B a chosen basis for $H_1(\Gamma,\mathbb{Z})$. Then

$$\mathcal{R} = K_0(\mathcal{V}_{\mathbb{C}})[T^2] \subset K_0(\mathcal{SV}_{\mathbb{C}})$$

where $T = [\mathbb{A}^{0|1}]$.

(square: double counting of fermion legs)

II

Graph insertions and singularities

Connes-Kreimer Hopf algebra of renormalization:

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

Milnor-Moore: Dual to the Lie algebra of insertions $\mathcal{L}_{CK}$, with Lie bracket

$$[\Gamma, \Gamma'] = \left(\sum_{\text{all vertices of } \Gamma} \Gamma \leftarrow \Gamma' \right) - \left(\sum_{\text{all vertices of } \Gamma'} \Gamma' \leftarrow \Gamma \right)$$

Kremnizer-Szczesny:

- $K_0(\mathbf{FGph}) \cong \mathbb{Z}[\mathcal{P}]$, $\mathcal{P}$ primitive graphs.

- $\mathbf{FGph}$ finitary abelian category and $\mathcal{L}_{CK}$ Ringel-Hall algebra associated to $\mathbf{FGph}$.

Problem — Lift CK insertion to the level of graph polynomials.

Bloch-Esnault-Kreimer:

$$\Psi_\Gamma = \Psi_\gamma \Psi_{\Gamma/\gamma} + f(A_1, \ldots, A_m)$$

where f is of deg $< h_1(\gamma)$ and $m = \#E(\Gamma/\gamma)$.

In joint work with *Christoph Bergbauer*, we found an explicit formula for the analogue of f in case of $\Gamma \leftarrow \Gamma'$, i.e., relate Ψ_Γ, $\Psi_{\Gamma'}$ and $\Psi_{\Gamma \leftarrow \Gamma'}$.

Notation/Steps:

- E_v: set of edges in $E_\Gamma \cup E_\Gamma^{\text{ext}}$ adjacent to vertex v of Γ.

- For e_1, $e_2 \in E_v$, $e_1 \sim e_2$ iff e_1, e_2 connected in $\overline{\Gamma - v}$.
  P_v: resulting partition of E_v.

- $P \leq P_v$: partition of P subordinate to P_v.

- γ_P: graph obtained by merging all the vertices $\partial s(q_1)$, . . . , $\partial s(q_n)$ for $\{q_1, \ldots, q_n\} \in P$ in γ. (s is the gluing map $E_v \to E_\gamma^{\text{ext}}$).

- Construction of graph $\Gamma^P(d)$:

  1. P induces a partition of V_{F_n} denoted as $P' = \partial b P$ (b: $E_v \to E_{F_n}^{\text{ext}}$ fixed bijection).

  2. d: spanning tree of F_n s.t. restriction to d of subgraphs of F_n connected.

  3. look at $\Gamma \leftarrow F_n$ and remove all edges of d which connect same cells of P'.

  4. shrink all edges of d which connect different cells of P'.

Def. — Let $P \leq P_v$ and $\Gamma^P(d)$ as above. Let

$$\tilde{\Psi}_{\Gamma^P}(d) := \sum_t \prod_{e \notin t} A_e$$

with the sum over all spanning trees t of $\Gamma^P(d)$ s.t. for each $Q \in P$ and e_1, $e_2 \in Q$, the path in t from e_1 to e_2 does not meet any edges in $E_v \setminus Q$.

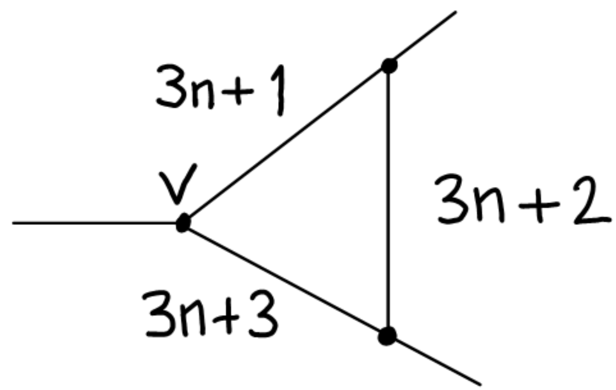
Rem. — $\tilde{\Psi}_{\Gamma^P}(d)$ is independent of d, so we can just write $\tilde{\Psi}_{\Gamma^P}$.

Theorem

$$\Psi_{\Gamma \leftarrow \gamma} = \Psi_\gamma \Psi_\Gamma + \sum_{0 \neq P \leq P_v} \Psi_{\gamma^P} \tilde{\Psi}_{\Gamma^P}$$

Example

$$\Gamma_n = \left\{ \right.$$

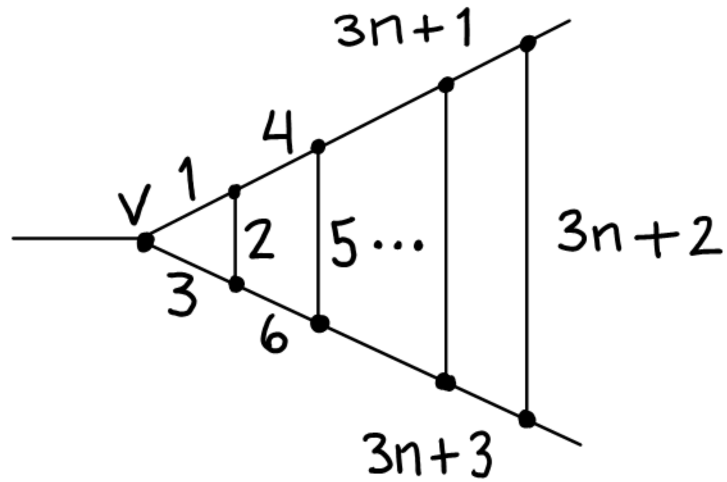


$$\Psi_{\Gamma_n} = A_{3n+1} + A_{3n+2} + A_{3n+3}$$

(Example cont'd)

Look at

$$\Gamma_n \leftarrow \cdots \leftarrow \Gamma_0 = \Bigg\{$$



Case $n = 1$:

$$\tilde{\Psi}_{\Gamma_0,\Gamma_1} = (A_1+A_3)A_2$$

$$\Psi_{\Gamma_1 \leftarrow \Gamma_0} = (A_1+A_2+A_3)(A_4+A_5+A_6)+(A_1+A_3)A_2$$

Case $n = 2$:

$$\tilde{\Psi}_{\Gamma_2,\Gamma_1 \leftarrow \Gamma_0} = ((A_1+A_2+A_3)(A_4+A_6)+(A_1+A_3)A_2)A_5$$

$$\Psi_{\Gamma_2 \leftarrow (\Gamma_1 \leftarrow \Gamma_0)} = \begin{cases} (A_1+A_2+A_3)(A_4+A_5+A_6)(A_7+A_8+A_9) \\ \quad +(A_1+A_3)A_2(A_7+A_8+A_9) \\ \quad +((A_1+A_2+A_3)(A_4+A_6)+(A_1+A_3)A_2)A_5 \end{cases}$$

Have a general closed formula as well.

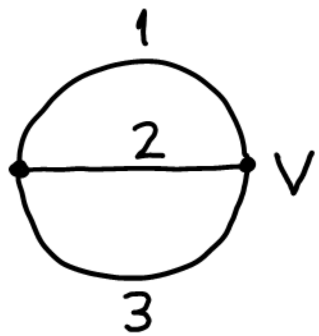
Consequences for singular loci:

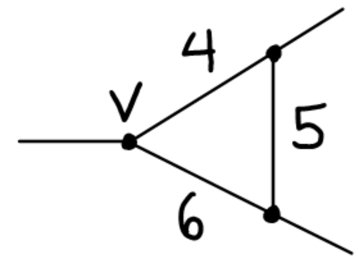
Corollary

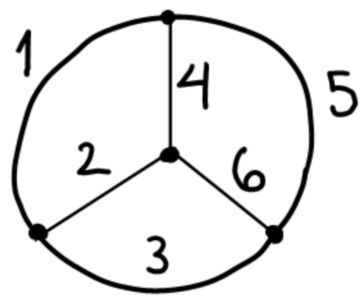
$$C\tilde{X}_{\gamma,\Gamma} \cap (CX_\gamma \cup CX_\Gamma) = CX_{\Gamma\leftarrow\gamma} \cap (CX_\gamma \cup CX_\Gamma)$$
$$\mathsf{Sing}\, C\tilde{X}_{\gamma,\Gamma} \cap (CX_\gamma \cap CX_\Gamma) = \mathsf{Sing}\, CX_{\Gamma\leftarrow\gamma} \cap (CX_\gamma \cap CX_\Gamma)$$
$$\mathsf{Sing}\, C\tilde{X}_{\gamma,\Gamma} \cap \mathsf{Sing}\, CX_\gamma = \mathsf{Sing}\, CX_{\Gamma\leftarrow\gamma} \cap \mathsf{Sing}\, CX_\gamma$$
$$\mathsf{Sing}\, C\tilde{X}_{\gamma,\Gamma} \cap \mathsf{Sing}\, CX_\Gamma = \mathsf{Sing}\, CX_{\Gamma\leftarrow\gamma} \cap \mathsf{Sing}\, CX_\Gamma$$

Her $C(\cdot)$ is the affine cone over the proj. space $(\cdot)$, Sing$(\cdot)$ denotes the singular locus, and $\tilde{X}_{\gamma,\Gamma} = \{\, \tilde{\Psi}_{\gamma,\Gamma} = 0 \,\}$.

Example

$\Gamma = \Big\{$ 

, $\gamma = \Big\{$ 

$\Gamma \leftarrow \gamma = \Big\{$ 

$$\Psi_\Gamma = A_1A_2 + A_1A_3 + A_2A_3$$

$$\Psi_\gamma = A_4 + A_5 + A_6$$

$$\tilde{\Psi}_{\gamma,\Gamma} = A_4A_5A_6 + A_1A_5(A_4+A_6) + A_2A_6(A_4+A_5) + A_3A_4(A_5+A_6)$$

1$^{\text{st}}$ term: $P_v = \{\{1,2,3\}\}$

2$^{\text{nd}}$ term: $P = \{\{1\},\{2,3\}\}$, etc.

$$\operatorname{Sing} CX_{\Gamma\leftarrow\gamma} = \left\{ \begin{array}{l} A_1A_4 + A_1A_5 + A_1A_6 + A_4A_6 = 0, \\ A_2A_4 + A_2A_5 + A_2A_6 + A_4A_5 = 0, \\ A_3A_4 + A_3A_5 + A_3A_6 + A_5A_6 = 0. \end{array} \right\}$$

Chern-Schwarz-MacPherson classes

- characteristic classes of singular varieties

- measures “how singular” varieties are

- gives Euler characteristic

Rem. — In general, it is the fact that graph hypersurfaces are singular that we get the motives associated to them to be *mixed* as opposed to pure.

Macaulay 2 computations (based on a program of *Aluffi*)

H: hyperplane class

Wheel with 3 spokes

Fulton class: $27H^5+6H^4+18H^3+9H^2+3H$

CSM class: $6H^5+12H^4+14H^3+9H^2+3H$

Milnor class: $-21H^5+6H^4-4H^3$

Half-open ladder

1^{st} *insertion:*

Fulton class: $6H^5+12H^4+14H^3+8H^2+2H$

CSM class: $5H^5+11H^4+13H^3+8H^2+2H$

Milnor class: $-H^5-H^4-H^3$

2^{nd} *insertion:*

Fulton class: $-162H^8+90H^7+54H^6+108H^5+90H^4+54H^3+18H^2+3H$

CSM class: $9H^8+34H^7+72H^6+96H^5+85H^4+50H^3+18H^2+3H$

Milnor class: $171H^8-56H^7+18H^6-12H^5-5H^4-4H^3$

Aluffi-Marcolli: For banana graphs with n parallel edges Γ_n, $n \geq 3$,

$$\chi(X_{\Gamma_n}) = n + (-1)^n. \qquad (*)$$

Conjecture — A formula very similar to $(*)$ holds true for the family of half-open ladders.

III

String vacua and computation theory

String vacua and computation theory

$\mathcal{C}$: configuration space of string vacua

- Landscape of string vacua, $|\mathcal{C}| \sim 10^{500}$ distinct solutions

- Question: is it possible to separate one point in $\mathcal{C}$ from another?
  Answer: *No!*

  - choice of average unification gauge group

  - moduli space of Ricci-flat metrics on a Calabi-Yau space

  - periods of Calabi-Yau and decidability

SM gauge group:

$$G_{SM} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)/(\mathbb{Z}/6\mathbb{Z})$$

Expect: at high energy ($\sim 10^{16}$ GeV), by running coupling constants, G_{SM} is a subgroup of some larger G.

There are representation-theoretic constraints on what G can be (assuming invariance under Poincaré group, etc.).

Choices for G:

$$\mathrm{SU}(5), \quad \mathrm{Spin}(10), \quad E_6, \quad (\mathrm{Spin}(6) \times \mathrm{Spin}(4))/(\mathbb{Z}/2\mathbb{Z})$$

Our requirements on G based on the fact that they describe the universe we live in!

Over $\mathcal{C}$, G can be wildly different!

Question — Can we describe whether an arbitrary $\mathbf{G}$ for a point in $\mathcal{C}$ contains G_{SM} or not?

Central parameter for statistical analysis:

average *rank* of the gauge group (w.r.t. suitable measure on $\mathcal{C}$)

This is expressed in terms of the complex moduli of the compactified space and configuration of D-branes mapping it.

Kumar-Wells: Fraction of all SUSY vacua that have gauge group rank R above SM gauge group rank R_{SM} is

$$\eta \sim \text{exp}\left(-\frac{R_{SM}}{\langle R\rangle}\right)$$

Gmeiner et al.: order of 10^{-9}.

Formulation

Fix a point in $\mathcal{C}$ and write the gauge group rank as $\mathbf{r}$ and the corresponding gauge group as $\mathbf{G}$. Imagine there exists a sequence of groups G_i with rank $G_i = \alpha_i$ such that either

- each individual G_i is a subgroup of $\mathbf{G}$ satisfying the set of conditions on $G \supset G_{SM}$

- or a product of G_i's satisfies the same.

Theorem — Given the rank $\mathbf{r}$ of $\mathbf{G}$ and the ranks of G_i being α_i, it is an NP-complete decision problem whether we can find a subsequence G_k, $1 \leq k \leq n$, $n = |I|$, such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \mathbf{r}$.

Proof is a direct application of subset-sum theorem and the *Grishko-H. Neumann* theorem on ranks of free products of groups:

$$\mathrm{rk}(G_1 * G_2) = \mathrm{rk}\, G_1 + \mathrm{rk}\, G_2.$$

One of our other results show a close formal similarity between the space

$$\text{Met}(M) = \text{Riem}(M)/\,\text{Diff}(M),$$

of Riemannian metrics on smooth closed manifolds modulo diffeo's, and the space

$$\text{Met}_J(X)$$

of metrics on a Calabi-Yau X with a fixed Kähler form J.

In case of Met(M) we know, following *Nabutovsky-Weinberger*, that it has a fractal structure. We expect the same for Met$_J(X)$ for arbitrary X. This implies that the problem of explicitly finding Ricci-flat metrics for a given X is computationally hard.

Periods and string vacua

$N = 2$ SUSY on CY 3-fold X

$$\varpi_i = \int_{\gamma^i} \Omega, \qquad \Omega\text{: hol. 3-form; } \gamma^i\text{: basis of homology cycles}$$

Fundamental period ϖ_0 can be explicitly computed for a large class of CY's.

Example — 1-param. family of quintic 3-folds given by $p(x,\psi) = \sum_{k=1}^{5} x_k^5 - 5\psi x_1 \cdots x_5$ with coordinates identified under the action of $G = (\mathbb{Z}/5\mathbb{Z})^3$:

$$\varpi_0(\psi) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}$$

with $|\psi| \geq 1$, $0 < \arg\psi < \frac{2\pi}{5}$.

After analytic continuation to $|\psi| < 1$:

$$\varpi_0(\psi) = -\frac{1}{5} \sum_{m=1}^{\infty} \frac{\Gamma(\frac{m}{5})(5\alpha^2\psi)^m}{\Gamma(m)\Gamma^4(1-\frac{m}{5})}$$

Idiosyncratic — Each point in $\mathcal{C}$ identified with a fundamental period.

Question — Can we distinguish points this way?

Answer — Unlikely, based on work of *Yoshinaga* and general expectations about periods.