Modeling Quantum Gravity: "A Noncommutative Perspective"

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December 18, 2010

Talk based on:

- Alain Connes, *Noncommutative geometry*, Academic Press, 1994.

- Alain Connes and Matilde Marcolli, *Anomalies, dimensional regularization and noncommutative geometry*, unpublished manuscript.

- Louis Crane and Lee Smolin, Renormalizability of general relativity on a background of spacetime foam, *Nucl. Phys. B* 267 (1986) 714–757.

- AR, *Anomalies and black holes as regulators in noncommutative geometry*, in preparation.

The riddle of quantum gravity

1. The modern theory of classical gravity is Einstein's general relativity $\mathbf{R} = -8\pi\mathbf{T}$.

2. All other three fundamental forces are *quantum gauge theories*.

3. Modern description of quantum gauge theories are as pQFT. These theories are *renormalizable*.

4. The issues related to renormalizability are precisely the ones that prevent Einstein gravity to be consistently quantized.

Feynman integrals and renormalization

1. Scattering amplitudes of processes given by integrals $U(\Gamma)$ associated to finite graphs Γ.

2. Each Γ encodes a term in a loop expansion of the Feynman path integral.

3. $U(\Gamma)$ is divergent for a generic Γ. $U(\Gamma)$ is made finite by regularization and then taking the polar part off.

4. Typical example from scalar QFT: for a 1-1 process ("vacuum diagram") in ϕ^3 theory, the integral is, for external momenta p and internal loop momenta k,

$$\int \frac{dk}{(p-k)^2 k^2}.$$

5. Dimensional analysis shows that this integral diverges as

$$\log k.$$

6. Render this integral finite by rescaling the couplings in the Lagrangian and cancel out the divergence by introducing through this rescaling similar divergences but with a - sign ("counterterms"). Often-used parameter governing the divergence: $D = n - \epsilon$ (with $n = 4$ in real-life).

7. These rescalings of the Lagrangian are governed by a certain function of the coupliing constants of the theory called the β-function. Examples of β-functions ("renormalization group flow"):

$$(\phi^3) \ \beta(g) \ = \ -\frac{g^3}{128\pi^3},$$

$$
\begin{aligned}
\text{(QED)}\ \beta(e) &= \frac{e^3}{12\pi^2} + O(e^5), \\
\text{(QCD)}\ \beta(g) &= -\frac{1}{48\pi^2}(33 - 2N_f)g^3.
\end{aligned}
$$

The jist of *renormalization* is that one can cancel all divergences of a theory by introducing a finite number of counterterms!

QED, QCD and the weak interactions are all renormalizable in this sense. Gravity is not!

There are several issues, some subtle and some not-so-subtle, involved in this:

1. Linearity: GR is a manifestly higly non-linear theory: by counting the number of components of the metric and curvature tensors, we have $10 = 6 + 4$ equations. Out of this, 6 are nonlinear PDEs and 4 are constraints on this. Quantum *mechanics* on the other hand is a linear theory.

2. Anomalies: problems with conformal invariance after quantization.

3. Bounded vs. unbounded Hamiltonians.

4. Divergence structure: using the structure of spacetime as a natural regulator.

Problems with a linear theory of quantum gravity (Penrose)

Assume a spin 2 graviton propagating on a flat Minkowski space M (with flat metric $h_{\mu\nu}$). Adding one gravition to M does not change the curvature but adding infinite number of them does!
Same as

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \text{ finite number of terms}$$

has a pole at $z = 0$ but the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{z^n} = \frac{1}{z-1}$$

has the pole shifted to $z = 1$.
This is the essential jist of the matter, considering the behavior of graviton propagators!

Anomalies: an under-appreciated story

Problem with *Weyl anomaly* with any *quantum* theory of gauge matter fields coupled with Einstein gravity with matter source T_{ij} (T_{ij} are from quantum fields so we talk about their expectation value $\langle T_{ij}(x)\rangle$).

1. Conformal transformations of the metric: $g_{ij}(x) \to \Omega^2(x) g_{ij}(x)$ ("angle-preserving transformations").

2. (Capper–Duff, 1960s) With DimReg $D = n - \epsilon$ in curved background, the Ward identity on the finite part about $\epsilon = 0$ of the self-energy insertion

   $$\Pi_{ijkl}(p) := \int d^n x e^{ipx} \langle T_{ij}(x) T_{kl}(0)\rangle|_{g_{ij}=\delta_{ij}}$$

   breaks down if we allow the conformal transformation of the metric. More precisely

Π^{i}_{ijk} of the finite part of the self-energy insertion $\neq 0$.

3. The Weyl anomaly is a trace anomaly defined as

$$\mathbf{w} := g^{ij}\langle T_{ij}(x)\rangle_{\text{reg}} - \langle g^{ij}T_{ij}(x)\rangle_{\text{reg}}$$

where $\langle - \rangle_{\text{reg}}$ denotes the (dim)regularized values.

4. The basic idea, put more simply, is that taking traces of the source and regularizing do not commute as operators and this "noncommutative difference" is the Weyl anomaly $\mathbf{w}$.

5. If one takes the traces of the source i.e. computes $g^{ij}\langle T^{ij}\rangle$ for different n and expands things out in terms of R and the

curvature tensor, the coefficients b_k that appear are the same as in the asympotic heat kernel expansions

$$\text{tr } e^{-t\Delta} \sim \sum_{k=0}^{\infty} B_k t^{k-\frac{n}{2}}$$

of appropriate differential operators Δ on M: $b_k = \int B_k d^n x$. These are called "the Schwinger-DeWitt coefficients".

6. In presence of gauge matter fields with N_S scalars, N_F spin-1/2 fermions and N_V vector fields, $\mathbf{w}$ depends only on two constants b_1 and b_2 with

$$\begin{aligned} b &= \frac{1}{120(4\pi)^2}(N_S + 6N_F + 12N_V), \\ b' &= -\frac{1}{360(4\pi)^2}(N_S + 11N_F + 62N_V). \end{aligned}$$

7. b and b' arise from the gravitational contribution only– the expansion of the trace

has terms with the external gauge fields coupled to curvature and even neglecting these terms, $\mathbf{w} \neq 0$.

8. Claim: this basic problem, of regularization not commuting with taking traces, i.e. the Weyl anomaly, is a quintessential quantum mechanical problem best understood in the language of Connes' noncommutative geometry.

9. Preliminary calculations within the framework of Connes–Marcolli bears testimony to this idea. Cf. also the 2009 Johns Hopkins thesis of Agarwala for a possible formulation using the renormalization bundle.

Spacetime regulators and quantum gravity (Crane–Smolin)

Basic idea: use a distribution of black holes $\rho(m)$ of mass $\leq M_p$ (the Planck mass) on a spacelike slice of M of mass between m and $m + dm$ to act as a natural regulator for integrals with graviton propagators.

1. Advantage: pure states map to mixed states in the process of evaporation of black holes (Hawking). Loss of quantum coherence.

2. Linearized Hamiltonian in this background is bounded from below, a requirement for any realistic quantum gravity.

3. Net effect: effective dimension of spacetime lowered from 4 to $4 - \epsilon$. This is like a

scaling dimension: in terms of 1/N expansion,

$$D(p^2) \sim \frac{1}{NM_\rho^\epsilon(-p^2)^{2-\frac{1}{2}\epsilon}}$$

as $-p^2 \to \infty$. Therefore we get a fractal distribution of black holes at a scale bounded from above by L_p.

4. Anomalous dimension of the graviton $\frac{1}{2}\epsilon$.

5. (Hawking et al.) A Euclidean four-manifold which is made by gluing together a large number of manifolds with the topology of the Euclidean Schwarzschild solution is a Euclidean description of a spacetime with a large number of black holes.

Noncommutative geometric approach

Basic set up– (1) distributions of black holes dense on M and (2) equivalent to gluing

$$M = M_1 \cup M_2 \cup \cdots \cup M_k.$$

This gives in terms of rings of functions on M_i:

$$C(M) = \bigotimes_{n=1}^{k} C(M_n).$$

Noncommutative description of this object (in the case of k in an "uncountably infinte" set.)

1. **Theorem:** Fibrations of the moduli space of Riemannian manifolds implies foliations in terms of spacelike slices Σ.

2. Have a functional analytic understanding of the ADM formalism of evolution of Σ.

3. The work of Crane–Smolin gives us a fractal description of a manifold M with the set of blowup singularities everywhere dense in M. This leads to a very good scaling behavior of the graviton propagator.

4. Fractals have a natural interpretation as noncommutative spaces: use these methods to put the results of Crane–Smolin in better mathematical setting.

5. Recent work of Denicola–Marcolli–Zainy al-Yasri provides an adequate framework.