Feynman motives (after Bloch-Esnault-Kreimer)

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References for this talk:

- Bloch, S., Esnault, H. and Kreimer, D.: On motives associated to graph polynomials, arXiv:math.AG/0510011.

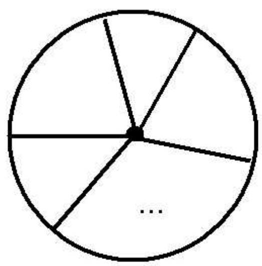
- Bloch, S.: Algebro-geometric aspects of Feynman graphs. Talk at IHÉS, March, 2006. Slides available at Bloch's Chicago website.

- Broadhurst, D. and Kreimer, D.: Knots and numbers in ϕ^4 theory to 7 loops and beyond, Int. J. Mod. Phys. C **6**, 519, 147-188.

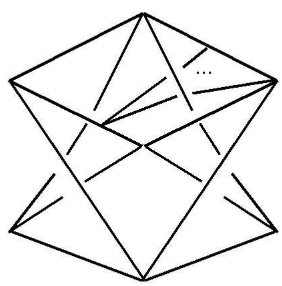
- Esnault, H.: Cohomology in the wheels-and-spokes case. Talk at IHÉS, June 2006.

Basic Problem

Computation of certain Feynman graphs give us zeta and multizeta values.
Examples from scalar field theory (with loop number l)



$\sim \zeta(2l-3)$



$\sim \zeta(5, 2l+3)$

Is there an algebro-geometric description of this phenomena in terms of motives and periods?

Graph hypersurfaces

Γ connected graph with E set of edges and V set of vertices of Γ. For each $e \in E$ attach a formal variable A_e and define the *Kirchoff polynomial* as

$$\Psi_\Gamma = \sum_T \prod_{e \notin T} A_e$$

Here T is a spanning tree of Γ.

Examples

- Γ with one vertex and n edges: $\Psi_\Gamma = \prod_{i=1}^n A_i$.

- Γ with two vertices and n edges between them: $\Psi_\Gamma = \sum_{i=1}^n A_1 A_2 \cdots \widehat{A}_i \cdots A_n$.

The *graph hypersurface* $X_\Gamma \subset \mathbb{P}^{\#E-1}$ is obtained from solving $\Psi_\Gamma = 0$.

Subgraph/subspace correspondence

Fix a connected graph Γ and let n be the number of edges of Γ. Let $h_1(\Gamma) :=$ rk $H_1(\Gamma, \mathbb{Z})$. We have

$$\text{Subgraphs } G \subset \Gamma \leftrightarrow \text{coordinate linear subspaces } L(G) \subset \mathbb{P}^{n-1}$$

Some properties:

- A linear subspace L is contained in X_Γ iff $h_1(G(L)) > 0$. Here $G(L)$ is the subgraph corresponding to linear subspace L.

- We have the identification
$$X_{\Gamma//G} = X_\Gamma \cap L(G)$$
Here G is a subgraph of Γ and $\Gamma \twoheadrightarrow \Gamma//G$ is the "modified" quotient. (Proof: Use the fact that $\Psi_{\Gamma//G} = \Psi_\Gamma|_{A_e=0, e\in G}$.)

Periods and Feynman quadrics

We work over $\mathbb{Q}$. Let $Q_i : q_i(Z_1, \ldots, Z_{2r}) = 0, 1 \le i \le r$ be homogeneous quadrics in $\mathbb{P}^{2r-1}$ Write $z_i = \frac{Z_i}{Z_r}$ and $\tilde{q}_i = \frac{q_i}{Z_{2r}^2}$ and define

$$\eta = \frac{\Omega_{2r-1}}{q_i \cdots q_r}$$

where

$$\Omega_{2r-1} := \sum_{i=1}^{2r} (-1)^i Z_i dZ_1 \wedge \cdots d\widehat{Z}_i \cdots \wedge dZ_{2r}$$

The *period* is defined as

$$P := \int_{\mathbb{P}^{2r-1}(\mathbb{R})} \eta = \int_{z_1,\ldots,z_{2r-1}=-\infty}^{\infty} \frac{dz_1 \wedge \cdots \wedge dz_{2r-1}}{\tilde{q_1} \cdots \tilde{q_r}}$$

For l some linear functional on $H \simeq \mathbb{Q}^4$ we define a rank 4 semidefinite quadratic form on $\mathbb{P}^{4n-1}$

$$q = q_i := (l_i^2, l_i^2, l_i^2, l_i^2)$$

Making a suitable choice of coordinates, we can write $q_l = Z_1^2 + \ldots + Z_4^2$.

Define a *Feynman quadric* by $Q_i : q = 0$. Let Γ be a graph with N edges and n loops. We define the *period* of Γ as

$$P(\Gamma) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \omega_\Gamma$$

where ω_Γ is of the form $\frac{d^{4n-1}x}{q_1 \cdots q_{2n}}$ and q_i's are the forms giving Feynman quadrics. (Here we have scaled $r = 2n$ in the previous definition of a period.)

Remarks:

- Γ is called *convergent* if $N > 2n$. It is called *log-divergent* if $N = 2n$.

- If Γ is log-divergent, then ω_Γ has poles along $\cup_i Q_i$..

Schwinger parametrization

Assume that all subgraphs of Γ are convergent. Then

$$P(\Gamma) = \frac{c}{\pi^{2n}} \int_{\sigma^{2n-1}(\mathbb{R})} \eta_\Gamma$$

where $\eta_\Gamma := \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2}$ and $\sigma^{2n-1}(\mathbb{R})$ is the locus of all points $[s_1, \ldots, s_{2n}]$ with $s_i \geq 0$.
Key benefit:
There is a tower

$$P = P_r \xrightarrow{\pi_{r,r-1}} P_{r-1} \xrightarrow{\pi_{r-1,r-2}} \ldots \xrightarrow{\pi_{2,1}} P_1 \xrightarrow{\pi_{1,0}} \mathbb{P}^{2n-1}$$

Let

$$\pi = \pi_{1,0} \circ \cdots \circ \pi_{r,r-1}$$

P_i: blowup of the strict transform of $L_i \subset X_\Gamma$. Then
$\pi^*(\eta_\Gamma)$ has no poles along the exceptional divisors from the blowups.

Intermezzo: Mixed Tate motives

Conjectural. category {MM}

- Cat. of semisimple objects of {MM} = $\{M\}_{\text{hom}}$.

- $\mathcal{M} \in$ Ob({M}) has a weigth filtration (increasing filtr. of $\mathbb{Q}$-vect. spaces) with graded factors in $\{M\}_{\text{hom}}$.

- There is cohomology realization $h^i(X)$ for X quasiproj.

- $\text{Ext}^i = 0$ for $i > 1$.

Subcategory of $\{\mathsf{MTM}\}/X$, X smooth projective scheme.

- $\{\mathsf{MTM}/\mathsf{Spec}\ \mathbb{Z}\}$ neutral Tannakian with forgetful fiber functor $\{\mathsf{MTM}/\mathsf{Spec}\ \mathbb{Z}\} \longrightarrow \mathsf{Vect}_{\mathbb{Q}}$.

- $\mathcal{M} \in \mathsf{Ob}(\{\mathsf{MTM}\}/X)$. Homs of $\{\mathsf{MTM}\}/X$ are compatible with wt. filtration and $\{\mathsf{MTM}\}/X \longrightarrow \mathsf{Gr}_{\bullet}^{W}\ \{\mathsf{MTM}\}/X$ is a exact tensor functor.

Feynman motives: Philosophy

General conjecture about {MTM/Spec $\mathbb{Z}$} due to Goncharov:

Let $\mathcal{M}$ be a MTM. Then the period of $\mathcal{M}$ is a multizeta value (more precisely, in $\mathbf{MZV}[\frac{1}{2\pi i}]$).

Expectation: associate to a Feynman graph Γ, a motive $[X_\Gamma]$.

If $[X_\Gamma]$ mixed Tate, then see if the period (obtained by Feynman rules) is MZV or not.

Remark: In general ∀Γ, $[X_\Gamma]$ is NOT mixed Tate.

Observation of Kontsevich:

$$\#X_\Gamma(\mathbb{F}_q) \in \mathbb{Z}[q] \Longrightarrow [X_\Gamma] \text{ mixed Tate}$$

Conjecture of Kontsevich:

$$\forall \Gamma, \#X_\Gamma(\mathbb{F}_q) \in \mathbb{Z}[q]$$

This was disproved by Belkale-Brosnan
$\rightsquigarrow$ $[X_\Gamma]$ in general not in $\{$MTM$\}$ for arbitrary Γ.
Question: Are there $[X_\Gamma]$ that have atleast a piece that is in $\{$MTM$\}$? Does Goncharov's conjecture hold true in these cases?

An appetizer

Let Γ be a graph with $\#E_\Gamma = 2h_1(\Gamma)$. $\Delta \subset \mathbb{P}^{2n-1}$ union of $2n$ coord. hyperplanes. Let $B := \pi^*\Delta$ and $Y \subset P$ strict transform of $X = X_\Gamma$.
Consider the cohomology groups

$$H_B^{2n-1}(P\backslash Y, B\backslash B \cap Y)$$

Then,

$$W_0 H_B \simeq \mathbb{Q}(0)$$

$$W_0 H_B \hookrightarrow H_B \xrightarrow{\int_{\tilde{\sigma}}} \mathbb{Q}$$

Want:

- $\mathrm{gr}_M^W H_B = \mathbb{Q}(-p)^{\oplus p}$
- rk 1- sub Hodge structure $\iota : \mathbb{Q}(-p) \hookrightarrow \mathrm{gr}_M^W H_B$ s.t. $\mathrm{im}(\eta_\Gamma) \in H_{\mathrm{DR}}$ spans $\iota(\mathbb{Q}(-p))_{\mathrm{DR}}$.

Feynman motives: Basic construction

By the Schwinger parametrization, we have

$$P_q(\Gamma) \in \mathbb{Q}^* \pi^{2n} P_g(\Gamma).$$

By the "tower argument",

$$\text{there is } \pi : P \longrightarrow \mathbb{P}^{2n-1} \text{ (birational)}$$

s.t.

$$\pi^*(\eta) \in \Gamma(\mathbb{P}^{2n-1}, \omega(2Y))$$

where Y is the strict transform of X.

$\rightsquigarrow \pi^*\eta \in \Gamma(P, \omega(2Y)) \longrightarrow H^{2n-1}_{\mathrm{DR}}(P \backslash Y, B \backslash B \cap Y)$

Again by the same argument,

$$\widetilde{\sigma^{2n-1}}(\mathbb{R}) \in H_{2n-1}(P \backslash Y, B \backslash B \cap Y)$$

$$= H^{2n-1}_B(P \backslash Y, B \backslash B \cap Y)$$

(from the fact that the cycle of int. disjoint from Y)

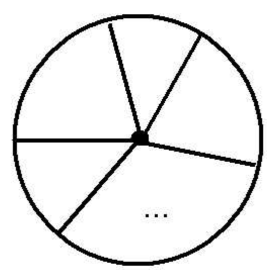
$\rightsquigarrow$ $P_q(\Gamma)$ is a period of $H^{2n-1}(P\backslash Y, B\backslash B\cap Y)$.
That is

$$\int_{\widetilde{\sigma^{2n-1}}} \pi^*\eta, \pi^*\eta \text{ alg. deRham form}$$

By physics computations (Broadhurst-Kreimer), we know that

$$P_q(\Gamma) \in \pi^{\mathbb{Z}}\mathbb{Q}^{\times}\zeta(p) \text{ for } p \text{ odd}$$

Specifically known for



Question: Is the corr. coh. group motivic?
Answer: Yes!

Main theorem of BEK

$$H^{2n-1}(\mathbb{P}^{2n-1}\backslash X_n) \simeq \mathbb{Q}(2n-3)$$

$$\text{equiv. } H^{2n-1}(X_n, \mathbb{Q})_{\text{prim}} \simeq \mathbb{Q}(-2)$$

where

$$H^{2n-1}(X_n, \mathbb{Q})_{\text{prim}}$$

$$:= \text{coker}(H^{2n-1}(X_n, \mathbb{Q}) \longrightarrow H^*(X_n, \mathbb{Q}))$$

Note the independence from n in the last statement!

“Cheat-sheet for proof”

Basically, three key tools:

- Artin vanishing: coh. groups vanish outside certain cones over a given hypsurf-vanishing depends on dim of coh group and that of the dim of the ambient proj space.

- Homotopy invariance

- Extensions of $\mathbb{Q}(0)$ to $\mathbb{Q}(p)$ through Borel's K-theory of number fields.

$$K_{2p-1}(\mathbb{Q}) \otimes \mathbb{Q} \simeq \mathbb{Q}$$

Show by first two that

$$H^i(\mathbb{P}^n \backslash V) = 0 \text{ for } i > a$$

where $V = \text{cone}(W \subset \mathbb{P}^a)$.

Then show that

$$H^{2n-5}(*, \mathbb{Q}) \simeq \mathbb{Q}(0)$$

where $*$ "$=$ some variety obtained from the gph. hypersurface by making a change of variables in Ψ_Γ and by a change of coordinates".

Now use the Borel isom. to get an extension of $\mathbb{Q}(0)$ to $\mathbb{Q}(p)$ to get the desired result for $p = 2n - 3$.