Periods, geometry and arithmetic in quantum fields and strings

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Why should I leave behind
my delight and entertainment,
those studies
that have swallowed up my mind?
– Robert Lowell *Jonathan Edwards in Western Massachusetts*

Preface

This dissertation lies at the intersection of several branches of mathematics– number theory, algebraic geometry and computation theory– with problems in modern theoretical physics, namely quantum theory, string- and M-theory. Despite the apparent lack of focus, what I have tried to do is to show how all of these areas are connected by the notion of a *period.* Introduced by Kontsevich and Zagier [**38**] in a seminal paper, periods are a very special kind of numbers lying between the algebraic closure of the rationals $\overline{\mathbb{Q}}$ and the complex numbers $\mathbb{C}$. They are typically obtained by integrating an algebraic de Rham form along a semi-algebraic set or a chain in singular homology. The modern theoretical framework to understand periods is through the theory of *motives*, a mother cohomology theory, if one wishes to think of it that way; avatars of these motives give rise to the plethora of cohomology theories encountered in arithmetic and algebraic geometry.

Recently, following the work of Belkale–Brosnan and Bloch–Esnault–Kreimer, a flurry of activity has been initiated in the direction of viewing the residue of a Feynman integral attached to a Feynman graph (or diagram, in the physics terminology) as periods related to mixed Tate motives– this is supposed to explain the mystery behind the physics computations started by Broadhurst–Kreimer in the mid nineties showing that large families of Feynman integrals evaluate to zeta and multiple zeta values. At the same time, there has been a lot of activity in trying to understand the Connes-Kreimer procedure for renormalization in terms of mixed Hodge structures (again, an avatar of motives.)

Independently the work of string theorists have shown how important periods are when discussing certain compactifications of Calabi–Yau manifolds as well as in mirror symmetry. The literature in this area is both deep and vast.

In this dissertation, I try to contribute certain results about periods, both in the context of Feynman graphs in quantum field theory and (more speculatively) in the context of string Landscape of vacua. In the latter case, we also discuss similarities between moduli of Riemannian spaces and that of Calabi-Yau complex manifolds. In terms of new results, we show/provide:

(1) A generalization of computation of log-divergent Feynman integrals in the case of theories with fermions; interestingly enough, in Chapter 2 we show that the resulting periods can be viewed as periods on supermanifolds. We hint at the possible relationship of this construction to periods of mirrors of rigid Calabi-Yau spaces. This is joint work with **Matilde Marcolli** and was published as *Supermanifolds from Feynman graphs* in J.Phys A: Math. Theor. 41 (2008) 315402 (21pp).

(2) The central work-horse of the motivic interpretation of Feynman periods is the Kirchhoff polynomial; setting this homogenous polynomial to zero

and thus obtaining a projective hypersurface gives us a basic geometric object to work with. In Chapter 3 we provide an explicit combinatorial formula for the Kirchhoff polynomial for a graph inserted into another graph. With this formula at hand we deduce several features of the singular loci of the graph hypersurfaces. One is reminded motives associated to Feynman graph hypersurfaces are of mixed type precisely because they are singular, so there is a natural interest in studying the singularity structure. We also take some first steps at understanding the characteristic classes of graph hypersurfaces using a computer program developed by Aluffi. This is joint work with **Christoph Bergbauer**, to be reported as a preprint soon.

(3) The Landscape of string vacua is perhaps the most hotly debated subject in cosmology and string theory these days. We use questions related to algorithmic undecidability and computational complexity theory to show that distinguishing points on the Landscape may be *conceptually harder* than previously expected. Right at the concluding section of this Chapter, we make "full circle" and return to questions about periods and their computational properties. This Chapter is based on my paper *Turing's Landscape: decidability, computability and complexity in string theory* available at the arXiv as arXiv:0909.1869v1 [hep-th].

We have also including an introductory chapter on the theory of motives for non-specialists, the justification for introducing a rather disproportionate amount of expository material being the relative lack of accessible accounts of (the technically simpler) aspects of motives most relevant for theoretical physics. This Chapter is available as an arXiv preprint *Motives: an introductory survey for physicists* (with an appendix by Matilde Marcolli which we omit from this Chapter), arXiv:0907.4046v2 [hep-th].

Acknowledgments

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Chapters 2 and 3 are joint work with Matilde Marcolli and Christoph Bergbauer respectively. I thank them for the exchange of ideas which made collaborations so pleasant. I also add that I am solely responsible if I have introduced errors in this exposition of our joint work.

I would especially like to thank the Fields Institute, Toronto where most of the actual writing was done and under rather difficult personal circumstances; in particular, I have been moved by the generosity of George Elliott and Eugene Ha with their time, encouragement and much more.

Lastly and perhaps most significantly, I would like to thank my advisor Matilde Marcolli for teaching me, sharing ideas with me and, most importantly, believing in me. Without her constant encouragement, none of this would have been possible.

Contents

CHAPTER 1

Motives: An introduction for non-specialists

The purpose of this Chapter is to familiarize an audience of physicists and other nonspecialists with some of the algebraic and algebro-geometric background upon which Grothendieck's theory of motives of algebraic varieties relies. There have been many recent developments in the interactions between high energy physics and motives, mostly within the framework of perturbative quantum field theory and the evaluation of Feynman diagrams as periods of algebraic varieties, though motives are beginning to play an important role in other branched of theoretical physics, such as string theory, especially through the recent interactions with the Langlands program, and through the theory of BPS states. We focus here mostly on the quantum field theoretic applications when we need to outline examples that are of relevance to physicists. In particular, we concentrate on motives, Feynman graphs and periods in the concluding section of this Chapter.

We describe the Grothendieck ring of varieties and its properties, since this is where most of the explicit computations of motives associated to Feynman integrals are taking place. We then discuss the Tannakian formalism, because of the important role that Tannakian categories and their Galois groups play in the theory of perturbative renormalization after the work of Connes–Marcolli. We then describe the background cohomological notions underlying the construction of the categories of pure motives, namely the notion of Weil cohomology, and the crucial role of algebraic cycles in the theory of motives. It is in fact mixed motives and not the easier pure motives are involved in the application to quantum field theory, due to the fact that the projective hypersurfaces associated to Feynman graphs are highly singular, as well as to the fact that relative cohomologies are involved since the integration computing the period computation that gives the Feynman integral is defined by an integration over a domain with boundary. We will not cover in this survey the construction of the category of mixed motives, as this is a technically very challenging subject, which is beyond what we are able to cover in this introduction. However, the most important thing to keep in mind about mixed motives is that they form *triangulated* categories, rather than abelian categories, except in very special cases like mixed Tate motives over a number field, where it is known that an abelian category can be constructed out of the triangulated category from a procedure known as the heart of a t-structure. For our purposes here, we will review, as an introduction to the topic of mixed motives, some notions about triangulated categories, Bloch–Ogus cohomologies, and mixed Hodge structures. We end by reviewing some the notion of motivic zeta function and some facts about motivic integration. Although this last topic has not yet found direct applications to quantum field theory, there are indications that it may come to play a role in the subject.

The contents of this Chapter covers many more things than those strictly needed for the rest of this dissertation. In particular we will not use any of the cohomological machinery described below; instead we would focus on classes in the Grothendieck ring of varieties. The main purpose here is to provide a context for much of the recent activity in this field.

1.1. The Grothendieck ring

1.1.1. Definition of the Grothendieck groups and rings. Recall the definition of the usual Grothendieck group of vector bundles on a smooth variety X:

DEFINITION 1.1 (exercise 6.10 of [**33**]). $K_0(X)$ is defined as the quotient of the free abelian group generated by all vector bundles (= locally free sheaves) on X by the subgroup generated by expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$ whenever there is an exact sequence of vector bundles $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. The group $K_0(X)$ can be given a ring structure via the tensor product.

We remark that isomorphism classes of vector bundles form an abelian monoid under direct sum; in fact, Grothendieck groups can be defined in a much more general way because of the following universal property:

PROPOSITION 1.2. *Let M be an abelian monoid. There exists an abelian group $K(M)$ and $\gamma : M \to K(M)$ a monoid homomorphism satisfying the following universal property: If $f : M \to A$ is a homomorphism into an abelian group A, then there exists a* unique *homomorphism of abelian groups $f_* : K(M) \to A$ such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & K(M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ A & = & A \end{array}$$

The proof of proposition 1.2 is very simple: construct a free abelian group F generated by M and let $[x]$ be a generator of F corresponding to the element $x \in M$. Denote by B the subgroup of F generated by elements of the form $[x+y]-[x]-[y]$ and set $K(M)$ to be the quotient of F by B. Letting γ to be the injection of M into F and composing with the canonical map $F \to F/B$ shows that γ satisfies the universal property.

One can show that projective modules over a ring A give rise to a Grothendieck group denoted as $K(A)$ (p. 138 of [**39**]). This is simply done by noting that isomorphism classes of (finite) projective A-modules form a monoid (again under the operation of direct sum) and taking the subgroup B to generated by elements of the form $[P \oplus Q] - [P] - [Q]$ for finite projective modules P and Q. We can refine $K(A)$ by imposing the following equivalence: P is equivalent to $Q \iff$ there exists free modules F, F' such that $[P \oplus F] = [Q \oplus F']$. Taking the quotient of $K(A)$ with respect to this equivalence relation gives us $K_0(A)$.

The explicit determination of K_0 (and its "higher dimensional" analogues) for a given ring A is the rich subject of algebraic K-theory. (A standard reference for algebraic K-theory is [**51**]; see also the work-in-progress [**57**].) For the simplest case when A is a field, we note that $K_0(A) \simeq \mathbb{Z}$; this immediately follows from the definition and the trivial fact that modules over a field are vectors spaces. As a non-trivial example, we have the following

EXAMPLE 1.3 (example 2.1.4 of [**57**]). Let A be a semisimple ring with n simple modules. Then $K_0(A) \simeq \mathbb{Z}^n$.

Through the theorem of Serre-Swan which demonstrates that the categories of vector bundles and finite projective modules are equivalent, we see that definition 1.1 arise very naturally from proposition 1.2. Furthermore, the generality of proposition 1.2 enables us construct Grothendieck groups and rings of (isomorphism classes of) other objects as well.

Let $\mathbf{Var}_k$ the category of quasi-projective varieties over a field k. Let X be an object of $\mathbf{Var}_k$ and $Y \hookrightarrow X$ a closed subvariety. Central to our purposes would be the following

DEFINITION 1.4. The Grothendieck ring $K(\mathbf{Var}_k)$ is the quotient of the free abelian group of isomorphism classes of objects in $\mathbf{Var}_k$ by the subgroup generated the expressions $[X] - [Y] - [X \setminus Y]$. The ring structure given by fiber product: $[X] \cdot [Y] := [X \times_k Y]$.

In fact one can generalize definition 1.4 to define the Grothendieck ring of a *symmetric monodial category.* Recall the following

DEFINITION 1.5 (defintion 5.1 of chapter II of [**57**]). A category $\mathbf{C}$ is called *symmetric monoidal* if there is a functor $\otimes : \mathbf{C} \to \mathbf{C}$ and a distinguished $\mathbb{I} \in$ Obj $(\mathbf{C})$ such that the following are isomorphisms for all $S, T, U \in$ Obj $(\mathbf{C})$:

$$\begin{aligned} \mathbb{I} \otimes S &\cong S \\ S \otimes \mathbb{I} &\cong S \\ S \otimes (T \otimes U) &\cong (S \otimes T) \otimes U \\ S \otimes T &\cong T \otimes S. \end{aligned}$$

Furthermore one requires the isomorphisms above to be *coherent*, a technical condition that guarantees that one can write expressions like $S_1 \otimes \ldots \otimes S_n$ without parenthesis without ambiguity[1]. (See any book on category theory for the precise definitions.)

REMARK 1.6. The category $\mathbf{Var}_k$ is symmetric monodial with the functor $\otimes$ given by fiber product of varieties and the unit object $\mathbb{I}$ given by $\mathbb{A}^0 =$ point.

An important example of a symmetric monoidal category is furnished by the category of finite dimensional complex representations of a finite group G (with the morphisms given by intertwinners between representations), denoted as $\mathbf{Rep}_{\mathbb{C}}(G)$.

EXAMPLE 1.7 (example 5.2.3 of [**57**]). $\mathbf{Rep}_{\mathbb{C}}(G)$ is symmetric monodial under direct sums of representations. Furthermore, $K_0(\mathbf{Rep}_{\mathbb{C}}(G)) \simeq R(G)$ generated by irreducible representations and where $R(G)$ is the representation ring of G.

[1]Note that we will sometimes call a symmetric monodial category an "ACU $\otimes$-category" and refer to the functor $\otimes$ as the *tensor functor*.

Note that for more general G (say an affine group scheme over an arbitrary base), the category $\mathbf{Rep}_{\mathbb{C}}(G)$ has much more structure that being merely symmetric monodial, namely it is a Tannakian category. We will turn to this rich subject in section 1.2.

1.1.2. The Grothendieck ring of varieties $K(\mathbf{Var}_k)$. Recall definition 1.4. The Grothendieck ring $K(\mathbf{Var}_k)$ has the following properties (cf. the review [**53**]):

(1) If X is a variety and U, V locally closed subvarieties in X, then
$$[U \cup V] + [U \cap V] = [U] + [V].$$
(2) If X is the disjoint union of locally closed subvarieties $X_1, \ldots, X_n$, then $[X] = \sum_{i=1}^n [X_i]$.
(3) Let C be a constructable subset of a variety X (that is, C is the disjoint union of locally closed subsets of X.) Then C has a class in $K(\mathbf{Var}_k)$.

In section 6 of [**41**], Marcolli proposes a category of Feynman motivic sheaves which involves viewing the Kirchhoff polynomial as a morphism $\Psi_\Gamma : \mathbb{A}^n \setminus X_\Gamma \to \mathbb{G}_m$ where X_Γ is viewed as an *affine* hypersurface obtained by setting $\Psi_\Gamma = 0$. (Here n is the number of edges of Γ and $\mathbb{G}_m$ is the multiplicative group.) As a setting for such relative questions, we introduce the following:

Definition 1.8 (Bittner, 2.1.1 of [**53**]). Let S be a variety over k. The ring $K(\mathbf{Var}_S)$ is the free abelian group generated by isomorphism classes $[X]_S$ (where X is a variety over S) modulo the relation $[X]_S = [X \setminus Y]_S + [Y]_S$ where $Y \subset X$ is a closed subvariety. The ring structure is induced by the fiber product of varieties.

Remark 1.9. We note the following properties of $K(\mathbf{Var}_S)$:

(1) $K(\mathbf{Var}_S)$ is a $K(\mathbf{Var}_k)$-module.
(2) There is an bilinear associative exterior product
$$K(\mathbf{Var}_S) \times K(\mathbf{Var}_T) \xrightarrow{\boxtimes} K(\mathbf{Var}_{S \times_k T}).$$
(3) Let $f : S \to S'$ be a morphism of varieties. It induces $f_! : K(\mathbf{Var}_S) \to K(\mathbf{Var}_{S'})$ and $f^* : K(\mathbf{Var}_{S'}) \to K(\mathbf{Var}_S)$ that are functorial with respect to $\boxtimes$: let $g : T \to T'$ be another morphism of varieties. Then $(f \times g)_!(A \boxtimes B) = f_!(A) \boxtimes g_!(B)$ and $(f \times g)^*(A \boxtimes B) = f^*(A) \boxtimes f^*(B)$.

Note that in case of $S = \mathrm{Spec} k$, the definition 1.8 is the same as definition 1.4.

Theorem 1.10 ([**6**], from [**53**]). *Let k a field of characteristic zero. Then $K(\mathbf{Var}_k)$ is generated by smooth varieties.*

The idea behind the proof is this (proof of proposition 2.1.2 of [**53**]): Set $d := \mathrm{dim} X$ and let $X \hookrightarrow X'$ for a complete variety X'. Writing $[X]+[Z] = [X']$ for some Z with $\mathrm{dim} Z \leq d-1$ and using Hironaka's theorem, we get $[\overline{X'}] = [X'] - ([C] - [E])$ where C is the smooth center of the blowup $\overline{X'}$ and E its exceptional divisor with $\mathrm{dim} C, \mathrm{dim} E \leq d-1$. We can write any arbitrary X as the disjoint union of smooth varieties in this way by inducting on d.

In $K(\mathbf{Var}_k)$, there are two "distinguished" classes, the class of a point $[\mathbb{A}^0]$ and the class of the affine line $\mathbb{A}^1 = \mathrm{Spec} k[x]$ denoted as $\mathbb{L}$. The standard cell decomposition of the projective space in terms of flags (see p.194 of [**29**]) can be lifted to $K(\mathbf{Var}_k)$ in terms of these classes:

$$\begin{aligned}[\mathbb{P}^n] = 1 + \sum_{i=0}^{n-1} \mathbb{L}^i &= \frac{1-\mathbb{L}^{n+1}}{1-\mathbb{L}} \\ &= \frac{(1+\mathbb{T})^{n+1}-1}{\mathbb{T}}\end{aligned}$$

where $\mathbb{T} := [\mathbb{A}^1] - [\mathbb{A}^0]$ the class of the torus $\mathbb{G}_m$.

In order to prove this decomposition, note that $[\mathbb{P}^1_k \setminus \mathbb{A}^1_k] = [\mathbb{P}^1_k] - [\mathbb{A}^1_k]$ since $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ is open with ∞ as the compliment. Induction gives the result since $\mathbb{P}^{n+1}_k \setminus \mathbb{A}^{n+1}_k \simeq \mathbb{P}^n_k$.

One can formally invert $\mathbb{L}$ in $K(\mathbf{Var}_k)$ to get the *Tate motive* denoted as $\mathbb{Q}(1)$. By $\mathbb{Q}(n)$ one means $\mathbb{Q}(1)\cdots\mathbb{Q}(1)$ (n-times) and $n \in \mathbb{Z}$ is called the *twist* of the Tate motive. One also sets as a matter of notation (as of now!) that $\mathbb{Q}(-1) := \mathbb{L}$.

A rather useful fact in the context of blowups of graph hypersurfaces is the following

Proposition 1.11 ([**53**]). *Let $f : X \to Y$ be a proper morphism of smooth varieties which is a blowup with the smooth center $Z \subset Y$ of codimension d. Then*

$$[f^{-1}(Z)] = [Z][\mathbb{P}^{d-1}].$$

As an example of decomposition of varieties into Lefschetz motives, we take a family of hypersurfaces in $\mathbb{P}^{n-1}$

$$X_{\Gamma_n} = \{(t_1, \ldots, t_n) | \Psi_{\Gamma_n} = t_1 \cdots t_n \Big(\frac{1}{t_1} + \cdots + \frac{1}{t_n}\Big) = 0\}.$$

The polynomial Ψ_{Γ_n} is the Kirchoff polynomial attached to a graph with two vertices and n number of parallel edges between them (the *banana graphs*); it arises in quantum field theory in the denominator of the Schwinger parametrized integral attached to the same graph. Then we have

Example 1.12 (Aluffi-Marcolli, theorem 3.10 of [**1**]). The class associated to the banana graph hypersurfaces X_{Γ_n} is given by

$$\begin{aligned}[X_{\Gamma_n}] &= \frac{(1+\mathbb{T})^n - 1}{\mathbb{T}} - \frac{\mathbb{T}^n - (-1)^n}{\mathbb{T}+1} - n\mathbb{T}^{n-2} \\ &= \frac{\mathbb{L}^n - 1}{\mathbb{L}-1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n(\mathbb{L}-1)^{n-2}.\end{aligned}$$

There is an alternative characterization of $K(\mathbf{Var}_k)$ due to Bittner, which is also sometimes useful in the context of Feynman graphs and motives.

Definition 1.13 (definition 2.3 of [**53**]). $K^{\mathrm{bl}}(\mathbf{Var}_k)$ is the abelian group generated by smooth complete varieties, subject to the conditions

(1) $[\emptyset] = 0$ and
(2) $[\mathrm{Bl}_Y X] = [X] - [Y] + [E]$ where Y is a smooth and complete subvariety of X and $\mathrm{Bl}_Y X$ is the blowup of X along Y with the exceptional divisor of the blowup E.

We need to show that the definitions 1.4 and 1.13 do coincide, following [**53**]. To do this we need to show that there exists a map $K(\mathbf{Var}_k) \to K^{\mathrm{bl}}(\mathbf{Var}_k)$ which is an isomorphism. This is constructed as an induced map from a map ϵ on varieties satisfying $e(X \setminus Y) = e(X) - e(Y)$. The map e is constructed in the following way: Let $\overline{X}$ be the smooth completion of a smooth connected variety X and $D = X \setminus \overline{X}$ a normal crossing divisor. Let us define the map $e(X) = \sum(-1)^l [D^{(l)}]_{\mathrm{bl}}$ with $D^{(l)}$ denoting the disjoint union of l-fold intersections of irreducible components of D. ($[-]_{\mathrm{bl}}$ denotes a class in $K^{\mathrm{bl}}(\mathbf{Var}_k)$.) The fact that $e(X)$ is independent of the choice of completion $\overline{X}$ follows from the weak factorization theorem. The fact that the $e(X) = e(X \setminus Y) + e(Y)$ is proved by choosing a smooth and complete $X \subset \overline{X}$ such that $D = \overline{X} \setminus X$ is simple normal crossing and the closure $\overline{Y}$ in X is smooth and normal-crossing with divisor D. ($D \cap Y$ is simple normal crossings divisor in Y.)

Yet another useful result, which can be useful in applications in the context of Feynman graphs, is the following.

Proposition 1.14 ([**53**]). *Let X be a smooth connected variety, $Y \subset X$ a smooth connected subvariety of X of codimension d. Let E be the exceptional divisor of the blowup $\mathrm{Bl}_Y X$ of X along Y. Then*

$$[\mathrm{Bl}_Y X] = [X] + \mathbb{L}[E] - \mathbb{L}^d [Y].$$

A consequence of proposition 1.14 is the notion of *Tate twist*: there is a ring involution

$$\mathbb{L} \mapsto \mathbb{L}^{-1} = \text{ and } [X] \mapsto \mathbb{L}^{-\dim X}[X] = \mathbb{Q}(1)^{\dim X}[X].$$

This is the true meaning of the Tate twist alluded above. In cohomological calculations, the product of $\mathbb{Q}(1)^{\dim X}$ and $[X]$ is replaced by a tensor product of a one dimensional rational vector space ("working mod torsion") raised to the dimension of the variety whose cohomology is being computed and a piece of the cohomology. In fact this correspondence between classes in $K(\mathbf{Var}_k)$ and cohomology is one important aspect of the theory of motives.

The reader may wonder what $K(\mathbf{Var}_k)$ explicitly is and how does it depend on k. The fact of the matter is that this is a very hard question though certain things are known. For example, we have

Theorem 1.15 (Poonen, theorem 1 of [**50**]). *Let k be a field of characteristic zero. Then $K(\mathbf{Var}_k)$ is not an integral domain.*

In fact when $k = \mathbb{C}$, we have

Theorem 1.16 (Larsen–Lunts, theorem 2.3 of [**40**]). *Let I be the ideal generated by the affine line, i.e., $I = \mathbb{L}$ and denote by SB the monoid of classes of stable birational complex varieties (the monoid structure coming from fiber products of varieties.) Then*

$$K(\mathbf{Var}_k)/I \simeq \mathbb{Z}[SB].$$

(Two varieties X and Y are *stably birational* if $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$ for $m, n \geq 0$.)

In Sahasrabudhe's thesis [**53**], theorem 1.16 is proved using Bittner's definition 1.13.

The main obstruction in obtaining results like theorems 1.15 and 1.16 in characteristic p is that those results depend crucially on the resolution of singularities and the weak factorization of birational morphisms, statements that are not yet known to hold true in characteristic p.

In Chapter 2 we generalize the construction of the Grothendieck ring of $\mathbf{Var}_k$ in a quite straightforward way for the category of *complex supermanifolds*.

1.2. The Tannakian formalism

1.2.1. Categorical notions. Most of the material in this section is based on [**20**] and Breen's survey of Saveedra-Rivano's thesis under Grothendieck [**52**]. In a nutshell, the main idea behind a Tannakian category is to equip an abelian category with a natural functor to vector spaces such that a fiber over each object in this category is a finite dimensional vector space.

Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category.

DEFINITION 1.17 (Internal homs). Let $X, Y \in \text{Obj}\ (\mathbf{C})$ and consider the functor

$$\begin{aligned} F : \mathbf{C} &\rightarrow \mathbf{Set}, \\ T &\mapsto \text{Hom}(T \otimes X, Y). \end{aligned}$$

If $F = \text{Hom}(-, K)$ where K is some object in $\mathbf{C}$, then define $\underline{\text{Hom}}(X, Y) =: K$.

REMARK 1.18. Explicitly, $\text{Hom}(T \otimes X, Y) = \text{Hom}(T, \underline{\text{Hom}}(X, Y))$. This spells out the fact that the functor F is *representable*.

EXAMPLE 1.19. Consider the category of all R-modules $\mathbf{Mod}_R$. This is an obvious tensor category with $\mathbb{I} = R$. In $\mathbf{Mod}_R$, $\underline{\text{Hom}}(X, Y) = \text{Hom}_{\mathbf{Mod}_R}(X, Y)$. To see this, define the functor F of definition 1.17 to be $T \mapsto \text{Hom}(T \otimes X, Y)$. Hom-tensor adjointness states $\text{Hom}_{\mathbf{Mod}_R}(T \otimes X, Y) = \text{Hom}_{\mathbf{Mod}_R}(T, \text{Hom}_{\mathbf{Mod}_R}(X, Y))$ and hence $F = \text{Hom}_{\mathbf{Mod}_R}(-, \text{Hom}_{\mathbf{Mod}_R}(X, Y))$ which, by definition, proves that $\underline{\text{Hom}}(X, Y) = \text{Hom}_{\mathbf{Mod}_R}(X, Y)$.

Let $\text{ev}_{X,Y} : \underline{\text{Hom}}(X, Y) \otimes X \rightarrow Y$ be the morphism corresponding to $\text{id}_{\underline{\text{Hom}}(X,Y)}$. To see that this correspondence does make sense, take $T = \underline{\text{Hom}}(X, Y)$ in definition 1.17 to get

$$\text{Hom}(\underline{\text{Hom}}(X, Y) \otimes X, Y) = \text{Hom}(\underline{\text{Hom}}(X, Y), \underline{\text{Hom}}(X, Y)).$$

Define the dual of an object X as $\hat{X} := \underline{\text{Hom}}(X, \mathbb{I})$. For example in $\mathbf{Mod}_R$ we have the following diagram

$$\begin{array}{ccc} \underline{\text{Hom}}(X, Y) \otimes X & = & \text{Hom}(X, Y) \otimes X \\ \downarrow{\scriptstyle \text{ev}_{X,Y}} & & \downarrow{\scriptstyle \text{ev}_{X,Y}} \\ Y & = & Y \end{array}$$

and $\text{ev}_{X,Y}$ is given by $f \otimes x = f(x)$, the usual evaluation map in $\mathbf{Mod}_R$. This shows that the abstract notion of evaluation makes sense.

DEFINITION 1.20 (Reflexive objects). Suppose we are given the following digram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(X,Y) & = & \underline{\mathrm{Hom}}(X,Y) \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle \mathrm{ev}_{X,\mathbb{I}}} \\ X \otimes \hat{X} & \xrightarrow{\mathrm{ev}_{X,\mathbb{I}}\circ\psi} & \mathbb{I} \end{array}$$

where X is an object in some tensor category. If $\mathrm{ev}_{X,\mathbb{I}} \circ \psi$ is an isomorphism, we say that X is reflexive. If all objects in a tensor category are reflexive, we call the category reflexive.

REMARK 1.21. We note that the above definition does correspond to our usual notion of reflexiveness: from the definition of internal hom,

$$\mathrm{Hom}(X \otimes \hat{X}, \mathbb{I}) = \mathrm{Hom}(X, \underline{\mathrm{Hom}}(\hat{X}, \mathbb{I})).$$

If $\mathrm{ev}_{X,\mathbb{I}} \circ \psi$ is an isomorphism, we get a correspondence

$$\mathrm{ev}_{X,\mathbb{I}} \circ \psi \longleftrightarrow X \xrightarrow{\sim} \hat{\hat{X}}$$

with $\mathrm{ev}_{X,\mathbb{I}} \circ \psi \in \mathrm{Hom}(X \otimes \hat{X}, \mathbb{I})$ and $X \xrightarrow{\sim} \hat{\hat{X}} \in \mathrm{Hom}(X, \underline{\mathrm{Hom}}(\hat{X}, \mathbb{I}))$.

Of course, not all categories are reflexive. For example, a one-line argument shows that $\mathbf{Mod}_R$ is not. (Take $R = \mathbb{Z}$ and consider $\mathbb{Z}/2\mathbb{Z}$.)

Putting all of these notions together, we have

DEFINITION 1.22 (Rigid tensor category). A tensor category $\mathbf{C}$ is said to be *rigid* if

(1) $\underline{\mathrm{Hom}}(X, Y)$ exists for every $X, Y \in \mathrm{Obj}\ (\mathbf{C})$.
(2) Functoriality of internal homs: All natural maps

$$\bigotimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i) \longrightarrow \underline{\mathrm{Hom}}\Big(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i\Big)$$

are isomorphisms.
(3) $(\mathbf{C}, \otimes)$ is a reflexive category.

We need a last assumption for the main definition, namely that a tensor category be *abelian*. We quickly review the definition: A category $\mathbf{A}$ is *abelian* [**25**] is

- $\mathrm{Hom}(X, Y)$ is an abelian group for all $x, y \in \mathrm{Obj}\ (\mathbf{A})$ and composition of morphisms is biadditive.
- There exists a zero object 0 such that $\mathrm{Hom}(0, 0) = \emptyset$.
- For all $X, Y \in \mathrm{Obj}\ (\mathbf{A})$, there exists an object $Z \in \mathrm{Obj}\ (\mathbf{A})$ and morphisms $\iota_X : X \to Z$, $\iota_Y : Y \to Z$, $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ with $\pi_X \circ \iota_X = id_X$, $\pi_Y \circ \iota_Y = id_Y$ and $\iota_X \circ \pi_X + \iota_Y \circ \pi_Y = id_Z$ and $\pi_X \circ \iota_Y = \pi_Y \circ \iota_X = 0$.
- For any $f \in \mathrm{Hom}(X, Y)$ there is a sequence $K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c}$ where K (resp. C) is the kernel of f (resp. cokernel of f) and such that $j \circ i = f$ and I is both kernel of c and cokernel of k.

With this last piece at hand, we have

DEFINITION 1.23 (Neutral Tannakian category). Let k be a field of arbitrary characteristic. A neutral Tannakian category over k is a abelian rigid tensor category $\mathbf{C}$ with a k-linear exact faithful functor called the *fiber functor* $\omega : \mathbf{C} \to \mathbf{Vect}_k$ with $\mathbf{Vect}_k$ the category of k-vector spaces[2].

To fix notation: a neutral Tannakian category would be denoted as a tuple $(\mathbf{C}, \otimes, \omega)$ where $\omega(X \otimes Y) = \omega(X) \otimes \omega(Y)$ from the definition 1.23. Faithfulness, exactness and k-linearity are the usual notions.

1.2.2. The structure of $\mathbf{Rep}_k(G)$. Let us come back to example 1.7 with G an *affine group scheme* instead, defined over k a field of arbitrary characteristic. Let V be a finite dimensional vector space over k. A representation of G is a morphism of affine schemes $G \times V \to V$, denoted as ρ. The category $\mathbf{Rep}_k(G)$ is defined in the following way.

- Objects of $\mathbf{Rep}_k(G)$ are representations (ρ, V).
- Morphisms between two representations are intertwiners. Let (ρ_1, V_1) and (ρ_2, V_2) be two representations. $\mathrm{Hom}_{\mathbf{Rep}_k(G)}((\rho_1, V_1), (\rho_2, V_2))$ consists of maps $f : V_1 \to V_2$ such that for every $g \in G$, the following diagram commutes:

$$\begin{array}{ccc} G \times V_1 & \xrightarrow{\rho_1} & V_1 \\ {\scriptstyle \mathrm{id}_G \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times V_2 & \xrightarrow{\rho_2} & V_2 \end{array}$$

The tensor structure is given by $\rho_1 \otimes \rho_2 : G \to \mathrm{Aut}(V_1 \otimes_k V_2)$, the unit object $\mathbb{I} : G \to \mathrm{Aut}(\mathbf{1})$ where $\mathbf{1}$ is the one-dimensional vector space with trivial G-action. (A more general construction of $\mathbf{Rep}_k(G)$ over R, a commutative ring with identity, ivolves V being a projective R-module of finite rank.)

By a general well-known argument, $\mathbf{Rep}_k(G)$ is equivalent to the category of commutative (though not necessarily cocommutative) Hopf algebras[3].

The main theorem along the lines of the classical Pontryajin duality is

THEOREM 1.24 (Tannaka-Krein theorem). *$\boldsymbol{Rep}_k(G)$ is a neutral Tannakian category with ω given by the forgetful functor. Furthermore* $\mathrm{Aut}^{\otimes}(\omega) \simeq G$.

The notation $\mathrm{Aut}^{\otimes}(\omega)$ denotes the natural transformations of the functor ω to itself preserving the tensor structure.

1.2.3. An equivalent definition. There is a definition of a neutral Tannakian category, due to Deligne [**20**], which is equivalent to definition 1.23. This definition has the benefit of being more explicit. I present it for the benefit of the reader who may want to gain a shift in perspective.

Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category over a field k with a unit object $\mathbb{I}$. (The k-linearity of the functor $\otimes$ is understood.) Furthermore *assume* $(\mathbf{C}, \otimes)$ to be abelian.

DEFINITION 1.25 (Rigid tensor category, alternate). The category $(\mathbf{C}, \otimes, \mathbb{I})$ is *rigid* if

[2]Some authors, for example Andre [**4**], demand an additional $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbf{Vect}_k$.

[3]This is a fact of vital importance in the Connes-Kreimer and Connes-Marcolli theories of renormalization as a Riemann-Hilbert problem.

Category	Unit	Dual
$\mathbf{Vect}_k$	$\mathbf{1}$	dual vector space $\hat{V}$
$\mathbf{Rep}_k(G)$ for G affine group scheme	k	$\hat{V}$ with induced G-action
Flat $\mathbb{C}$-vector bundles on X	$\underline{\mathbb{C}}_X$	$\widehat{\mathcal{E}} = \mathrm{Hom}(\mathcal{E}, \underline{\mathbb{C}}_X)$
Connections on $\mathbb{G}_m$	$\mathbb{G}_a$	$(\widehat{\mathcal{E}}, \widehat{\nabla})$

TABLE 1. Examples of rigid tensor categories

(1) $\mathrm{End}(\mathbb{I}) = \mathrm{Hom}(\mathbb{I}, \mathbb{I}) \simeq k$,
(2) For all $X \in \mathrm{Obj}\ (\mathbf{C})$ there exist objects $\hat{X} \in \mathrm{Obj}\ (\mathbf{C})$ and morphisms $\delta : \mathbb{I} \to \hat{X} \otimes X$ and $\mathrm{ev} : X \otimes \hat{X} \to \mathbb{I}$ such that

$$X \xrightarrow{\mathrm{id}\otimes\delta} X \otimes \hat{X} \otimes X \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} X,$$
$$\hat{X} \xrightarrow{\delta\otimes\mathrm{id}} \hat{X} \otimes X \otimes \hat{X} \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} \hat{X}.$$

We now *define* the internal hom to be $\underline{\mathrm{Hom}}(X, Y) := \hat{X} \otimes Y$ for all $X, Y \in \mathrm{Obj}\ (\mathbf{C})$. One can verify, upon making the identification $\mathrm{Hom}(X, Y) = \mathrm{Hom}(X, Y \otimes \mathbb{I})$ that this coincide with definition 1.17. The dual is a functor $\mathbf{C} \to \mathbf{C}$ given by $\hat{(-)} = \underline{\mathrm{Hom}}(-, \mathbb{I})$. It exists if and only if $\underline{\mathrm{Hom}}(X, -)$ exists and $\underline{\mathrm{Hom}}(X, \mathbb{I}) \otimes Y \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, Y)$. Deligne's definition of a neutral Tannakian category is the same as definition 1.23, namely, an abelian rigid tensor category with a fiber functor $\omega : \mathbf{C} \longrightarrow \mathbf{Vect}_k$, the functor being exact, faithful, k-linear and preserving the tensor structure.

The main theorem, in any case, is a stronger formulation of theorem 1.24, variously attributed to Deligne, Grothendieck and Saavedra-Rivano:

THEOREM 1.26. *Let $\boldsymbol{T}$ be a neutral Tannakian category over a field k. Then $\mathrm{Aut}^{\otimes}(\omega)$ is an affine group scheme over k and we have the equivalence of categories*

$$\boldsymbol{T} \simeq \boldsymbol{Rep}_k\Big(\mathrm{Aut}^{\otimes}(\omega)\Big),$$

where $\boldsymbol{Rep}_k(-)$ is the category of k-linear representations. Furthermore, if $\boldsymbol{T} \simeq \boldsymbol{Rep}_k(G)$ for some affine group scheme G, then there exists an exact, faithful, k-linear tensor functor ω such that

$$G \xrightarrow{\sim} \mathrm{Aut}^{\otimes}(\omega).$$

The functor ω is representable, following the remark 1.18. Ultimately, the rigidity features guarantee the group structure. In fact, the definition of ω in terms of "functor-of-points" suggests we look at ω with **Set** as the target category; this provides an interpretation of the action of the group $\mathrm{Aut}^{\otimes}(\omega)$ as a fundamental group, a theme we take up next.

1.2.4. Fundamental groups of schemes. Tannakian categories can also be used as a language to understand a beautiful theory developed by Grothendieck, Artin and others, of the relationship between fundamental groups of étale coverings of a scheme and Galois groups. The canonical reference for this material is SGA I [**30**].

DEFINITION 1.27 (Étale covering of a scheme). Let Y be a scheme over k. An étale covering $Y \to X$ is an affine morphism given by $A \to B$ with $X = \mathrm{Spec}A$ and $Y = \mathrm{Spec}B$, A and B algebras, and such that

(1) B is flat over A,
(2) $\mathrm{Der}_A B = 0$ and
(3) B is finite over A.

EXAMPLE 1.28. Let X be a smooth variety over $\mathbb{C}$. Saying that $Y \to X$ is an étale covering over X is the same as $Y_{\mathbb{C}} \to X_{\mathbb{C}}$ is a covering in the usual sense.

EXAMPLE 1.29. Let $X = \mathrm{Spec}k$. Then we have the equivalence of sets (in fact of categories!):

$$\{\text{connected étale coverings of X}\} \simeq \{\text{finite seperable field extensions of } k\}.$$

For a given and fixed scheme X, étale coverings form a category with morphisms $Y \to Y'$ and such that the following diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

commutes for étale coverings $Y \to X$ and $Y' \to X$. Denote this category as $\textbf{ÉtCov}_X$. Given a geometric point $\bar{x} \in X$, define a functor

$$\begin{aligned} \omega : \textbf{ÉtCov}_X &\longrightarrow \textbf{Set}, \\ (Y \xrightarrow{\pi} X) &\mapsto \pi^{-1}(\bar{x}). \end{aligned}$$

REMARK 1.30. In case of the universal covering $\tilde{X} \to X$ has the property that $\mathrm{Hom}(\tilde{X}, X) = \omega(Y)$ does not exist but

$$\omega(Y) = \varprojlim_i \mathrm{Hom}(X_i, Y),$$

This motivates our next definition.

DEFINITION 1.31 (Fundamental group).

$$\pi_1(X, \bar{x}) := \varprojlim_i \mathrm{Aut}_X(X_i).$$

EXAMPLE 1.32. Let X be a complex variety. Then $\pi_1(X) = \pi_1 \widehat{(X_{\mathbb{C}}, x_{\mathbb{C}})}$.

More interestingly,

EXAMPLE 1.33. Let $X = \mathrm{Spec}k$. Then $\pi_1(X) = \mathrm{Gal}(\bar{k}/k) =: G_k$.

In fact, we have, following definition 1.31:

THEOREM 1.34 (Grothendieck). *The category of finite étale schemes over k is equivalent to the category of finite sets with continuous G_k-action.*

In fact, in a Tannakian category with the fiber functor ω, the theorem 1.34 can be understood as saying that the fundamental group acts as automorphisms of ω. To see this, view the category of flat vector bundles over a scheme X as being equivalent to the category of representations of $\pi_1(X, x)$. Furthermore, the category is recovered by studying the category of representations of $\pi_1(X,x)$ using theorem 1.26.

I remark that theorem 1.34 is a special case of the general *Grothendieck-Galois correspondence* which states that the category of finite étale k-schemes is equivalent to the category of finite sets with continuous G_k-action.

1.2.5. The function-sheaf correspondence. The function-sheaf correspondence is a profound application of constructibility to obtain all "interesting" functions on a space over an arbitrary base ring in terms of certain sheaves on it. As such, it also connects with the *Geometric Langlands program.* In this subsection I scratch the surface, closely following the notes of Sug Woo Shin [**55**].

Fix a prime l and let K be a finite extension of $\mathbb{Q}_l$. Denote by $\mathcal{O}_K$ the ring of integers of K. Let X be a connected scheme over k with char $k \neq l$. Let X_{et} be an étale site of X. Call the étale sheaf $\mathcal{G}$ *locally constant* is $f|_U$ is locally constant for an étale covering U of X. The sheaf G is said to be *constructible* is X is constructible (i.e. can be written as disjoint union of locally closed subschemes of X) and $\mathcal{G}$ is such that it defines a locally constant sheaf, finite on each strata of X.

Definition 1.35 (locally constant l-adic sheaf). An l-adic sheaf on X_{et} is a projective system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of constructible sheaves $\mathcal{F}_n$ such that $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isomorphism $\mathcal{F}_{n+1} \otimes (\mathbb{Z}/l^n) \simeq \mathcal{F}_n$. An l-adic sheaf is *locally constant* if each $\mathcal{F}_n$ is so.

One defines a *category of K-sheaves* as a category whose objects are constructible $\mathcal{O}_K$-sheaves (= constructible sheaves of $\mathcal{O}_K$-modules) and with

$$\mathrm{Hom}_{K-\mathrm{sheaves}}(\mathcal{F}\otimes K, \mathcal{G}\otimes K) := \mathrm{Hom}_{\mathcal{O}_K-\mathrm{sheaves}}(\mathcal{F},\mathcal{G}) \otimes_{\mathcal{O}_K} K.$$

By taking a limit over l of the categories of K-sheaves, we obtain the category of $\overline{\mathbb{Q}}_l$-sheaves. Locally constant $\overline{\mathbb{Q}}_l$-sheaves are limits of locally constant $\mathcal{O}_K$-sheaves.

Definition 1.36 (l-adic local system). A locally constant $\overline{\mathbb{Q}}_l$-sheaf is called an l-adic local system.

Let $\overline{x} \in X$ be a geometric point. Let $\mathcal{E}$ be an l-adic local system on X. The stalk of $\mathcal{E}$ is defined as the direct limit of stalks of K-sheaves. These are, in turn, defined in the following way: for $\mathcal{F}$ a $\mathcal{O}_K$-sheaf on X, the stalk at $\overline{x}$ is $\varprojlim_n (\mathcal{F}_n)_{\overline{x}}$ where $(\mathcal{F}_n)_{\overline{x}}$ are the usual stalks on the étale site. Denote the stalk of $\mathcal{E}$ at $\overline{x}$ as $\mathcal{E}_{\overline{x}}$. We say that the local system $\mathcal{E}$ has *rank r* if $\mathcal{E}_{\overline{x}}$ contains r-copies of $\overline{\mathbb{Q}}_l$.

Theorem 1.37. *Let $\overline{x} \in X$ be a geometric point of a finite connected scheme X. The functor $\mathcal{F} \to \mathcal{F}_{\overline{x}}$ makes the category of l-adic local systems on X equivalent to the category of continuous representations of $\pi_1(X,\overline{x})$ on finite dimensional $\overline{\mathbb{Q}}_l$ vector spaces.*

Compare this with theorem 1.34: in both cases, action of the automorphisms of the fiber functor are interpreted in terms of "nice" (= étale!) geometric categories.

We end this discussion by noting a special case of the function-sheaf correspondence: let H be a (connected, seperated) commutative group scheme of finite type over $\mathbb{F}_{p^n}$ with $p \neq l$ and let $\mathcal{E}$ be an l-adic local system on H. Let m^* denote the pull-back of the multiplication map $m : H \times_{\mathbb{F}_{p^n}} H \to H$. We call $\mathcal{E}$ an

l-adic *character sheaf* if $m^*(\mathcal{E}) \simeq p_1^*(\mathcal{E}) \otimes p_2^*(\mathcal{E})$. Here p_1 and p_2 are the projection morphisms $H \times_{\mathbb{F}_q} H \to H$.

THEOREM 1.38 (Function-sheaf correspondence). *We have a natural bijection of sets*

$$\mathrm{Hom}_{\boldsymbol{AbGp}}(H(\mathbb{F}_q), \overline{\mathbb{Q}}_l) \leftrightarrow \{l\text{-}adic\ character\ sheaves\ on\ H\}.$$

REMARK 1.39. A basic fact: any commutative group scheme over a field k is always an extension of an abelian variety by an affine group k-scheme (= a commutative k-Hopf algebra.) Therefore, we loose nothing by following the notation and thinking of H as an abelian variety. For example, the reader may think of H as the Picard group of a curve, to fix ideas.

The proof of theorem 1.38 uses the functoriality of the action of the absolute Frobenius[4] on the stalks of l-adic local system $\mathcal{E}$ (cf. [**55**] for a nice sketch of the proof.)

A speculation: Upon a fixed identification $\overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, one way to think of theorem 1.38 is[5]: all character maps ϕ of a given commutative Hopf algebra $\mathbf{H}$ arise as appropriate l-adic character sheaves on $\mathbf{H}$. It remains a very interesting project to actually implement this philosophy when $\mathbf{H}$ is the Connes-Kreimer Hopf algebra of Feynman graphs[6] ([**15**] [**16**]) and ϕ a Feynman rule that assigns a generally divergent projective integral to a given Feynman graph and understanding its relationship with the Arapura motivic sheaves and renormalization as outlined in Marcolli's paper [**41**].

1.3. Weil cohomology

This section presents the notion of a Weil cohomology, a set of functorial properties any good cohomology theory of smooth projective varieties should satisfy. The section very closely follows the first parts of Kleiman's article in [**36**].

1.3.1. Main definition. Let k (resp. K) be fields of arbitrary characteristic (resp. zero characteristic.) Let $\mathbf{SmProj}_k$ be the category of smooth projective schemes over k with smooth scheme maps as morphisms. Let **GrVect** be the category of $\mathbb{Z}$-graded anticommutative K-algebras with K-algebra maps that preserve gradings as morphisms.

Suppose $V \in$ Obj (**GrVect**). Therefore $V := \bigoplus_{i \in \mathbb{Z}} V_i$. Set $V_0 = K$. A Weil cohomology is a contravariant functor

$$H^* : \mathbf{SmProj}_k \longrightarrow \mathbf{GrVect}$$

with $H^*(X) = \bigoplus_{n \in \mathbb{Z}} H^i(X)$ that satisfies the following properties:

[4] Let X be a (connected) scheme over $\mathbb{F}_{p^n}$. By the "absolute Frobenius" we mean an $\mathbb{F}_{p^n}$-morphism which is the identity on X as a topological space and is the map $x \mapsto x^p$ on the structure sheaf $\mathcal{O}_X$.

[5] There are several subtleties involving base change and defining the absolute Frobenius, so the reader should take this statement with a grain of salt.

[6] In the Connes-Kreimer theory, the coproduct of $\mathbf{H}$ gives a recursive formula for the factorization of loops in the prounipotent complex Lie group $G(\mathbb{C}) := \mathrm{Hom}(\mathbf{H}, \mathbb{C})$. The Birkhoff factorization of ϕ as algebra homomorphisms, needed for the counterterms, satisfy the Rota-Baxter identity.

1.3.1.1. *Finiteness.* Each $H^i(X)$ has finite dimension and $H^i(X) = 0$ unless $0 \leq i \leq 2\mathrm{dim}X$.

1.3.1.2. *Poincaré duality.* There are two equivalent versions. The first version goes as follows: Let $r := \mathrm{dim}X$. For each such X, there is an isomorphism

$$H^{2r}(X) \xrightarrow{\sim} K$$

and a nondegenerate pairing

$$H^i(X) \times H^{2r-i}(X) \xrightarrow{\sim} K.$$

The second version of the Poincaré duality goes as follows:

$$\widehat{H^i(X)} \simeq H^{2r-i}(X),$$

where $\widehat{H^i(X)} = \mathrm{Hom}(H^i(X), K)$.

1.3.1.3. *Künneth formula.* Let $X \times_k Y$ be the fiber product of two objects X and Y of $\mathbf{SmProj}_k$. Consider the following diagram of projections

$$\begin{array}{ccc} X \times_k Y & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Y}\downarrow & & \\ Y & & \end{array}$$

The projections induce the isomorphism

$$H^*(X) \otimes_K H^*(Y) \simeq H^*(X \otimes Y).$$

NB: since we are working over fields, the usual "Tor" term is absent.

1.3.1.4. *Cycle maps.* Let $C^i(X)$ denote the free abelian group generated by closed irreducible subschemes of X of codimension i. In a Weil cohomology, there should be a group homomorphism

$$\gamma_X^i : C^i(X) \longrightarrow H^{2i}(X)$$

satisfying the following properties:

(1) [Functoriality] There are two functorial conditions on cycle maps. The first ("pull back") goes as follows: Let $r := \mathrm{dim}X$. Let $f : X \to Y$ be a morphism of $\mathbf{SmProj}_k$. Let $C^i(X)$ and $C^i(Y)$ denote the group of cycles of codimension i on X and Y respectively. The following diagram commutes:

$$\begin{array}{ccc} C^i(Y) & \xrightarrow{f^*} & C^i(X) \\ {\scriptstyle \gamma_Y^i}\downarrow & & \downarrow{\scriptstyle \gamma_X^i} \\ H^{2i}(Y) & \xrightarrow{f^*} & H^{2i}(X) \end{array}$$

(The top map is a pull-back on cycles.)

The second functorial condition ("push forward") goes as follows: Let $s := \mathrm{dim}Y$. Now consider the pull back map defined above: if

$$H^{2i}(Y) \xrightarrow{f^*} H^{2i}(X),$$

then by Poincaré duality, we obtain the push-forward map

$$H^{2r-2i}(X) \xrightarrow{f_*} H^{2s-2i}(Y)$$

such that the following diagram commutes:

$$\begin{array}{ccc} C^{r-i}(X) & \xrightarrow{f_*} & C^{s-i}(X) \\ \gamma_X^{r-i}\downarrow & & \downarrow\gamma_Y^{s-i} \\ H^{2r-2i}(X) & \xrightarrow{f_*} & H^{2s-2i}(Y) \end{array}$$

(The map on the top denotes push-forward on cycles.)

(2) [Multiplicativity] The map $\gamma^{i+j}_{X\times_k Y} = \gamma^i_X(Z) \otimes_K \gamma^j_Y(W)$. We see, by applying the definition of the cycle map and the Künneth formula that

$$\begin{array}{rcl} Z \times_k W \in C^{i+j}(X \times_k Y) & \overset{\gamma^{i+j}_{X\times_k Y}}{\longrightarrow} & H^{2(i+j)}(X \times_k Y) \\ & \overset{\sim}{\longrightarrow} & \bigoplus_i H^{2(i+j)-l} H^{2(i+j)-l}(X) \otimes_K H^l(Y) \end{array}$$

so the multiplicativity axiom makes sense.

(3) [Calibration] Let P be a point. Then the cycle map $\gamma_P : C^0 \to H^0(P)$ is the same as the inclusion of $\mathbb{Z}$ in K.

1.3.1.5. *Lefschetz theorems.* There are two versions of the Lefschetz theorems, one "weak" and the other "strong". Let $h : W \to X$ be the inclusion of a smooth hyperplane section W of some smooth projective scheme X of dimension r.

(1) [Weak Lefschetz] The induced map $h^* : H^i(X) \to H^i(W)$ is an isomorphism for $i \leq r-2$ and an injection for $i = r-1$.

(2) [Hard Lefschetz] Define the *Lefschetz operator*

$$\begin{array}{rcl} L : H^i(X) & \to & H^{i+2}(X) \\ Lx & \mapsto & x \cdot \gamma^1_X(W). \end{array}$$

Then for $i \leq r$, $L^{r-i} : H^i(X) \longrightarrow H^{2r-i}(X)$ is an isomorphism.

In summary: a Weil cohomology theory over k is a contravariant functor from the category $\mathbf{SmProj}_k$ to the category $\mathbf{GrVect}$ with a nondegenerate pairing satisfying a Poincaré duality, with a Künneth formula, with cycle maps that are functorial with respect to pull-backs and push-forwards and satisfying the Lefschetz theorems.

1.3.2. Some properties of cycle maps. There are two important properties of cycle maps in the context of pure motives.

1.3.2.1. *Intersection pairing and cup products.* Let Z and W be two properly intersecting cycles in X and let $Z \cap W$ denote the intersection pairing. Let $\Delta : X \to X \times X$ be the diagonal map. Then $\gamma_X(Z \cap W) = \gamma_{X\times X} \circ \Delta^*(Z \times W)$ where Δ^* denotes the pull-back of $Z \times W$. We knew from the functoriality of cycle maps that this equals $\Delta^* \circ \gamma_{X\times X}(Z \times W) = \Delta^*(\gamma_X(Z) \otimes \gamma_X(W))$ by the multiplicativity of the cycle maps. So we get

$$\Delta^*(\gamma_X(Z) \otimes \gamma_X(W)) = \gamma_X(Z) \cdot \gamma_X(W)$$

by applying pull-back again.

1.3.2.2. *Correspondences as operators.* Let $r := \dim X$. Observe from the definitions that

$$\begin{aligned} H^i(X \times Y) &= H^i(X) \otimes H^i(Y) \\ &= \widehat{H^{2r-i}(X)} \otimes H^i(Y) \\ &= \mathrm{Hom}(H^{2r-i}(X), K) \otimes H^i(Y) \\ &= \mathrm{Hom}(H^{2r-i}(X), H^i(Y)). \end{aligned}$$

Therefore, we can view each element of $H^i(X \times Y)$ as an *operator* from $H^{2r-i}(X)$ to $H^i(Y)$. Such operators are called *correspondences* for the obvious reasons of section 1.4.

1.3.3. The Standard Conjectures. The original formulation is in Grothendieck's Tata lecture [**31**]. I follow Murre's lecture [**47**].

Rescalling the indices in the hard Lefschetz theorem, the isomorphism can be equivalently written as $L^{r-i} : H^{r-j}(X) \to H^{r+j}$ for $r = \dim X$. Using the map L we can define a unique linear operator Λ which makes the following: For $0 \leq j \leq r-2$ the diagram

$$\begin{array}{ccc} H^{r-j}(X) & \xrightarrow{L^j} & H^{r+j}(X) \\ \Lambda\downarrow & & \downarrow L \\ H^{r-j-2}(X) & \xrightarrow{L^{j+2}} & H^{r+j+2}(X) \end{array}$$

commutes, and for $2 \leq j \leq r$

$$\begin{array}{ccc} H^{r-j+2}(X) & \xrightarrow{L^{j-2}} & H^{r+j-2}(X) \\ L\uparrow & & \uparrow \Lambda \\ H^{r-j}(X) & \xrightarrow{L^j} & H^{r+j+2}(X) \end{array}$$

the diagram commutes. We have

Conjecture 1.40 (B(X)). *The operator Λ is algebraic. That is*

$$\Lambda(Z) = \gamma^*_{X \times X}(Z) \textit{ for all } Z \in \mathrm{CH}^i(X \times X) \otimes \mathbb{Q}.$$

By the hard Lefschetz theorem, we have the isomorphisms $L^{r-i} : H^i(X) \to H^{2r-i}(X)$. Let

$$P^i(X) := \ker\{L^{r-i-1} : H^i(X) \to H^{2r-i+2}(X)\}$$

be the set of *primitive elements* of $H^i(X)$. Let $x, y \in C^i(X) \cap P^{2i}(X)$ for $i \leq r/2$.

Conjecture 1.41 (Hdg(X)). *Consider the pairing*

$$(x, y) \mapsto (-1)^i \langle L^{r-2i}(x), y \rangle$$

where $\langle -, - \rangle$ denotes the cup product. This pairing is positive definite.

It is known that conjecture 1.41 is true for étale cohomology in characteristic zero. (The fact that étale cohomology is a Weil cohomology and that, in particular, the hard and weak Lefschetz theorems hold, requires some work to show.)

Conjectures 1.40 and 1.41 taken together imply the Weil conjectures, among other things. One cares about the standard conjectures because they capture something intrinsic about the cycles independent of the Weil cohomology that the cycle map maps to. For example, we have the following, due to Grothendieck:

THEOREM 1.42. *Assume conjectures 1.40 and 1.41. Then the Betti numbers* $\dim H^i(X)$ *is independent on the choice of* X*.*

1.4. Classical motives

As noted in the introduction, one of the central goal of theory of motives is *linearization* of the category of algebraic varieties (or schemes) over an arbitrary base field (or ring). In fact, what we are after is obtaining something stronger than additivity (which is the "coarsest" form of linearization)– we want an abelianization of the category of algebraic varieties! One way to abelianize the category of varieties is to replace morphisms in this category by *correspondences*.

1.4.1. Correspondences of curves. By a curve, we mean a nonsingular variety of dimension 1. Let us start with the case of correspondences of curves over the complex numbers (classical theory due to Castelnuovo and Severi).

DEFINITION 1.43 (section 2.5 of [**29**]). Let C and C' be two curves. A *correspondence* of degree d between C and C' is a holomorphic map

$$T : C \to C', p \mapsto T(p)$$

where $T(p)$ a divisor of degree d on C. Furthermore, the *curve of correspondence* between C and C' is given by the curve

$$D = \{(p,q) | q \in T(p)\} \subset C \times C'.$$

Let $D \subset C \times C'$ be a curve. Then the correspondence associated to D is defined by

$$T(p) = \pi_p^*(D) \in \mathrm{Div}(C')$$

where $\pi_p^* : C' \to C \times C'$ is given by $q \mapsto (p,q)$.

REMARK 1.44. The "converse" part of definition (1.43) states that we should view correspondences as formal linear combinations of all 0-dimensional subvarieties on C'.

1.4.2. Equivalence relations on algebraic cycles. All through we will work with smooth projective varieties over a field k. We denote the category of such objects as $\mathbf{SmProj}_k$. Furthermore, most of the time, we'd restrict ourselves to such varieties that are also irreducible. Much of the material in this section (and the next) is from the excellent reviews by Murre [**47**] and Scholl [**54**]. Recall

DEFINITION 1.45. An *algebraic cycle* Z on X of codimension i is a formal linear combination of closed irreducible subvarieties Z_α of codimension i; thus,

$$Z = \sum_\alpha n_\alpha Z_\alpha, n_\alpha \in \mathbb{Z}.$$

Algebraic cycles (or cycles, for short) of a given codimension i in a given smooth projective X form an abelian group with repect to formal sums. Denote such a group as $\mathcal{Z}^i(X)$. As it stands, $\mathcal{Z}^i(X)$ is very "big", so we want to obtain a smaller group by taking a quotient of $\mathcal{Z}^i(X)$ by equivalence relations on cycles.

There are several equivalence relations that one can impose on cycles, but we'd restrict ourselves to the following.

(1) (Rational equivalence) Let Z be a cycle of a fixed codimension i on X irreducible of dimension d. We say Z is rationally equivalent to zero and write $Z \sim_{\text{rat}} 0$ if there exists (Y_α, f_α) with Y_α irreducible of codimension $i-1$ and f_α a rational function on Y_α such that $\sum_\alpha v_{Y_\alpha}(f_\alpha)Y_\alpha = Z$. (Here v denotes the valuation of the rational function.)

(2) (Homological equivalence) Once again, let Z be a cycle of codimension i. Let $H^*_B(X, \mathbb{Q})$ be the Betti cohomology of X. Then there exists a map (in fact a group homomorphism!) $Z \mapsto [Z]$ where $[Z]$ is a cohomology class in $H^{2i}_B(X, \mathbb{Q})$. We say that Z is homologically equivalent to zero and write $Z \sim_{\text{hom}} 0$ if the image of Z under this map vanishes. Furthermore, cycle maps are general gadgets that associate i-cycles to $2i$-cohomology classes in any good cohomology theory (not just Betti) and statisfying certain functorialilty conditions, a notion formalized by Weil cohomology, see 1.3.1. Homological equivalence is a notion that is to hold true in *any* (and perhaps more strongly in *all*) of these good cohomology theories.

(3) (Numerical equivalence) Let Z as in above. We say Z is numerically equivalent to zero and write $Z \sim_{\text{num}} 0$ if the intersection number $\#(Z \cap Z') = 0$ for all Z' cycles of the same codimension as Z. (To spell things out a bit more: the definition says that $\#(Z_\alpha \cap Z'_\beta) = 0$ for all α, β and where $Z = \sum_\alpha n_\alpha Z_\alpha$ and $Z' = \sum_\beta n_\beta Z'_\beta$.)

It can seen that the relations given above are indeed equivalence relations. From now on, we would write $\sim$ to mean any of three relations above unless we fix the equivalence relation, in which case we would specify it as such.

Definition 1.46. Fix a smooth projective variety X. The group

$$C^i_\sim(X) := \mathcal{Z}^i(X)/\mathcal{Z}^i_\sim(X)$$

where $\mathcal{Z}^i_\sim(X)$ denotes the subgroup of cycles identified under the equivalence relation $\sim$. When X is irreducible and of dimension d, $C_\sim(X) := \oplus_{i=0}^d C^i_\sim(X)$. When $\sim = \sim_{\text{hom}}$, $C^i_\sim(X)$ is called the *Chow group* and is denoted as $CH^i(X)$. Notation: we write $C_\sim(X)_\mathbb{Q}$ to mean $C_\sim(X) \otimes \mathbb{Q}$.

Remark 1.47. Notice that (1) is just a generalization of the notion of divisors on curves and that of linear equivalence. The relation (3) is trickier because the intersection pairing $Z \cap Z'$ may not be always defined. In this case, one resorts to various *moving lemmas*. Perhaps the most problematic is the equivalence relation (2) since the definition is based on a choice of a cohomology theory; however assuming one of the implications of the Standard conjectures along with Jannsen's result help solving this problem (see theorem 1.55.)

An important observation is

$$C^i_{\sim_{\text{rat}}}(X) \subseteq C^i_{\sim_{\text{hom}}}(X) \subseteq C^i_{\sim_{\text{num}}}(X) \subset C^i_\sim(X)$$

going from the finest to the coarsest equivalence relations on cycles. An extremely important conjecture in the theory of classical motives is

CONJECTURE 1.48 (Fundamental Conjecture D(X)).

$$C^i_{\sim_{\mathrm{hom}}}(X) = C^i_{\sim_{\mathrm{num}}}(X).$$

The conjecture 1.48 follows from the Standard Conjectures of Grothendieck B(X) (conjecture 1.40) and Hdg(X) (conjecture 1.41).

1.4.3. Pure motives. Just like in the case of curves (see definition (1.43), we view correspondences on a smooth projective variety as formal linear combinations of closed irreducible subvarieties, i.e, as elements in *group of correspondences* $C_\sim(X \times Y)_\mathbb{Q} =: \mathrm{Corr}_\sim(X, Y)$.

Let $f \in \mathrm{Corr}_\sim(X, Y)$ and $g \in \mathrm{Corr}_\sim(Y, Z)$. The composition of correspondences is given by

$$g \bullet f = \mathrm{pr}_{XZ}((f \times Z) \cdot (X \times g)).$$

A emphprojector $p \in \mathrm{Corr}^0_\sim(X, X)$ is an idempotent $p \bullet p = p$.

DEFINITION 1.49 (the category of effective pure motives). Fix an equivalence relation on cycles $\sim$. The category of effective pure motives over a field k is denoted as $\mathbf{Mot}^+_\sim(k)$ has

- Objects: (X, p) where $X \in \mathrm{Obj}\ (\mathbf{SmProj}_k)$ and p a projector.
- Morphisms: Let $M = (X, p)$ and $N = (Y, q)$. Then $\mathrm{Hom}_{\mathbf{Mot}^+_\sim(k)}(M, N) := q \bullet \mathrm{Corr}^0_\sim(X, Y) \bullet p$.

Elements of $\mathrm{Hom}_{\mathbf{Mot}^+_\sim(k)}(M, N)$ are simply the composition of correspondences $X \to X \to Y \to Y$. By modifying the definition 1.49 to allow for Tate twists, we get the category of virtual pure motives over k.

DEFINITION 1.50 (the category of virtual pure motives). The category of virtual pure motives over a field k is denoted as $\mathbf{Mot}_\sim(k)$ has

- Objects: (X, p, m) where X and p as above and $m = \mathrm{dim} X$.
- Morphisms: Let $M = (X, p, m)$ and $N = (Y, q, n)$ (and $n = \mathrm{dim} Y$). Then $\mathrm{Hom}_{\mathbf{Mot}_\sim(k)}(M, N) := q \bullet \mathrm{Corr}^{n-m}_\sim(X, Y) \bullet p$.

Some trivial and distinguished pure motives: $\mathbf{0} := (\mathrm{Spec} k, \mathrm{id}, 0)$, the motive of a point, $\mathbf{L} := (\mathrm{Spec} k, \mathrm{id}, -1)$ the Lefschetz motive and $\mathbf{T} := (\mathrm{Spec} k, \mathrm{id}, -1)$ the Tate motive.

1.4.4. The motives functor. Recall that if $\psi : Y \to X$ is a scheme morphism, then the *graph* of ψ, $\Gamma_\psi := (\psi \times id_Y) \circ \Delta_Y$ where id_Y and Δ_Y are respectively the identity and diagonal maps on Y. That is, the graph is simply the composition of maps in the sequence

$$Y \xrightarrow{\Delta_Y} Y \times Y \xrightarrow{\psi \times id_Y} X \times Y.$$

The transpose Γ^t_ψ is obtained by exchanging the factors of $X \times Y$.

DEFINITION 1.51. The functor

$$\mathfrak{m}_\sim : \mathbf{SmProj}^{op}_k \to \mathbf{Mot}_\sim(k)$$

is defined in the following way: $\mathfrak{m}_\sim(X) := (X, \Delta_X, 0)$ and $\mathfrak{m}_\sim(\psi) = \Gamma_\psi^t : \mathfrak{m}_\sim(Y) \to \mathfrak{m}_\sim(X)$ and where Γ_ϕ^t is the transpose of the graph of $\psi : X \to Y$ a morphism of smooth projective schemes.

Let $e \in X$ be a point. Take $\pi_0 = e \times X$ and $\pi_{2d} = X \times e$ where $d := \dim X$ and X is irreducible. Write $\mathfrak{m}_\sim^0(X) := (X, \pi_0, 0)$ and $\mathfrak{m}_\sim^{2d}(X) := (X, \pi_{2d}, 0)$. We can show that

$$\mathfrak{m}_\sim^{2d}(X) \simeq (\mathrm{Spec} k, \mathrm{id}, -d)$$

and use this to show the fundamental fact

$$\mathbf{L} \simeq \mathfrak{m}_\sim^2(\mathbb{P}^1, \mathbb{P}^1 \times e, 0)$$

where $\mathbb{P}^1$ is the projective line over k and e a point in $\mathbb{P}^1$.

The category of (virtual) pure motives $\mathbf{Mot}_\sim(k)$ has the following properties.

- $\mathbf{Mot}_\sim(k)$ is an additive category. That is, $\mathrm{Hom}_{\mathbf{Mot}_\sim(k)}(M, N)$ are abelian groups and $\oplus$ exists: $M \oplus N := (X \amalg Y, p \amalg q, m)$ for $\dim X = \dim Y = m$.
- $\mathbf{Mot}_\sim(k)$ is a pseudoabelian category; it is additive with a well-defined image of p.
- There is a tensor structure on $\mathbf{Mot}_\sim(k)$:

$$M \otimes N := (X \times Y, p \times q, m + n).$$

- There is a multiplicative structure on $\mathbf{Mot}_\sim(k)$:

$$m_X : \mathfrak{m}_\sim(X) \otimes \mathfrak{m}_\sim(X) \simeq \mathfrak{m}_\sim(X \times X) \xrightarrow{\mathfrak{m}_\sim(\Delta)} \mathfrak{m}_\sim(X).$$

Let k be a field and $\overline{k}$ be its algebraic closure. Fix an equivalence relation $\sim$ on the cycles of X. Let M be a virtual pure motives of a smooth projective variety X, i.e, $M = \mathfrak{m}_\sim(X)$.

Definition 1.52 (Realization of a pure motive). Define the realization of a motive M as the functor

$$\mathbf{Mot}_\sim(k) \xrightarrow{\mathbf{real}} \mathbf{GrVect} \mathfrak{m}_\sim(X) \mapsto H^*(X_{\overline{k}}, \mathbb{Q})$$

for a Weil cohomology functor $H^*(-, \mathbb{Q})$. We say that a motive can be realized if there exists at least one Weil cohomology for which the functor **real** is exact and faithful for all adequate relations $\sim$.

The reader may set $H^*(X_{\overline{k}}, \mathbb{Q})$ as algebraic de Rham cohomology as to fix ideas.

Grothendieck's original conception of motives was such that the following diagram of functors commute *for all adequate equivalence relations $\sim$ and all Weil cohomologies $H^*(X, \mathbb{Q})$*:

$$\begin{array}{ccc}
\mathbf{Mot}_\sim(k) & = & \mathbf{Mot}_\sim(k) \\
{\scriptstyle \mathfrak{m}_\sim(-)} \uparrow & & \downarrow {\scriptstyle \mathbf{real}(-)} \\
\mathbf{SmProj}_k^{op} & \xrightarrow{H^*(-, \mathbb{Q})} & \mathbf{GrVect}
\end{array}$$

There are three categories of motives of particular interest to us: the category of *Grothendieck motives* when the equivalence relation is homological equivalence $\mathbf{Mot}_{\mathrm{hom}}(k)$, the category of *Chow motives* when the equivalence relation is homological equivalence $\mathbf{Mot}_{\mathrm{rat}}(k)$ and the category of *numerical motives* when the equivalence relation is numerical equivalence $\mathbf{Mot}_{\mathrm{num}}(k)$. We have the following "fundamental theorem of pure motives":

THEOREM 1.53. *$\mathbf{Mot}_{hom}(k)$ can be realized for all Weil cohomologies and is neutral Tannakian with the fiber functor **real**.*

This is the reason why number theorists often use the term "motive of X" when they mean "the l-adic cohomology of X with $\mathbb{Q}_l$-coffecients and a continuous G_k action on it". (The action of G_k is through the motivic Galois group of X, cf. section 1.4.5.)

Recall the yoga of Tannakian formalism of section 1.2. Deligne defines the dimension of an object in a rigid tensor category $\mathbf{C}$ in the following way:

DEFINITION 1.54 (dimension). Let $\mathbb{I}$ be the unit object in $\mathbf{C}$. Let $M \in \mathrm{Obj}\ (\mathbf{C})$ be an arbitrary object. Let $f \in \mathrm{End}M$ and trace of f on M is defined by the composition of maps

$$\mathbb{I} \xrightarrow{\delta} \widehat{M} \otimes M \xrightarrow{t} M \otimes \widehat{M} \xrightarrow{\mathrm{ev}} \mathbb{I}$$

and denote as $\mathrm{tr} f_M$. (The map t exchanges the factors in $\widehat{M} \otimes M$.) The dimension of M is $\underline{\mathrm{dim}} M := \mathrm{tr} 1_M$.

A result of Deligne show that dimension is always positive: $\mathbf{C}$ rigid tensor $\Longleftrightarrow$ for all $M \in \mathrm{Obj}\ (\mathbf{C})$, $\underline{\mathrm{dim}} M \in \mathbb{N}$.

Furthermore for $\mathbf{Mot}_{\mathrm{hom}}(k)$, Deligne shows that this definition of dimension coincides (through the Weil conjectures) with our usual notion of cohomological dimension:

$$\underline{\mathrm{dim}} M = \mathrm{dim} H^*(X, \mathbb{Q}) := \oplus_i \mathrm{dim} H^i(X, \mathbb{Q})$$

where $H^i(X, \mathbb{Q})$ is a realization of the motive M. The Standard Conjectures imply that $\mathrm{dim} M$ is independent of the choice of the Weil cohomology.

THEOREM 1.55 (Jannsen). *The category $\mathbf{Mot}_{num}(k)$ is semisimple and abelian.*

1.4.5. Motivic Galois groups. The Tannakian formalism along with theorem 1.53 gives

$$\mathbf{Rep}_k(\mathrm{Aut}^{\otimes} H_B^*(-, \mathbb{Q})) \xrightarrow{\sim} \mathbf{Mot}_{\mathrm{hom}}(k) \xrightarrow{\mathbf{real}(-)} \mathbf{GrVect} \circlearrowleft G_{\mathrm{mot}}$$

where $G_{\mathrm{mot}} := \mathrm{Aut}^{\otimes} H_B^*(-, \mathbb{Q})$ is the *motivic Galois group*. It acts on the image of the realization functor and is a proalgebraic group. (It is proreductive assuming Jannsen's theorem 1.55 and the Conjecture D(X) 1.48.)

If $\mathbf{Mot}_{\sim}(k)$ is generated by $\mathfrak{m}_{\sim}(\mathrm{Spec} E)$ where E is a finite extension of k then $G_{\mathrm{mot}} = G_k$. If $\mathbf{Mot}_{\sim}(k)$ is generated by the Lefschetz motive $\mathbf{L}$, then $G_{\mathrm{mot}} = \mathbb{G}_m$.

1.5. Mixed motives

Pure motives are motives of smooth projective varieties. In most physical applications of the theory of motives, most notably in the case of the motives of hypersurfaces associated to Feynman graphs in perturbative quantum field theory, working with smooth projective varieties is too restrictive. In fact, one knows that the projective hypersurfaces obtained from the parametric form of Feynman integrals are typically singular, and this already leaves the world of pure motives. Moreover, since the parametric Feynman integral (see the appendix) is computed over a domain of integration with boundary, what is involved from the motivic point of view is really a relative cohomology of the hypersurface complement, relative to a normal crossings divisor that contains the boundary of the domain of integration. This is a second reason why it is mixed motives and not pure motives that are involved, since these are the natural environment where long exact cohomology sequences and relative cohomologies live. Unfortunately, from the mathematical point of view, the theory of mixed motives is far more complicated than that of pure motives. At present, one only has a triangulated category of mixed motives, with various equivalent constructions due to Voevodsky, Levine, and Hanamura. Only in very special cases, such as mixed Tate motives over a number field, it is possible to construct and abelian category. While we are not going to give any details on the construction of the triangulated category of mixed motives, we recall the relevant cohomological properties and some properties of their analytic counterpart, mixed Hodge structures. We also recall some preliminary notions about triangulated categories to aid the reader in her further study.

1.5.1. Derived and triangulated categories. [7] This material is taken from Dimca [**22**]. Let **Ab** be an abelian category. Let $C(\mathbf{Ab})$ be the category of complexes in **Ab**. (The reader unfamiliar with homological algebra should immediately set **Ab** to be the category of R-modules $\mathbf{Mod}_R$.) $C(\mathbf{Ab})$ contains three important full subcategories $C^\bullet(\mathbf{Ab})$:

(1) $C^+(\mathbf{Ab})$ of complexes bounded on the left:
$$\cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow A^{-1} \longrightarrow A^0 \longrightarrow \cdots$$
(2) $C^-(\mathbf{Ab})$ of complexes bounded on the right:
$$\cdots \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \cdots$$
(3) $C^b(\mathbf{Ab})$ the full subcategory of complexes bounded both on the right and the left.

Let $X^\bullet, Y^\bullet$ be two complexes in $C^\bullet(\mathbf{Ab})$. Call the morphism $u : X^\bullet \to Y^\bullet$ a *quasi-isomorphism* if at the level of cohomology $H^k(u) : H^k(X^\bullet) \to H^k(Y^\bullet)$ is an isomorphism for all k.

Define a *shift automorphism* on complexes $T : C^\bullet \to C^\bullet$ as
$$(X[n])^r := X^{n+r},$$
$$d^s_{T(X^\bullet)} := -d^{s+1}_{X^\bullet}$$
for the complex
$$A^\bullet : \cdots \longrightarrow A^{m-1} \xrightarrow{d^{m-1}} A^m \xrightarrow{d^m} A^{m+1} \xrightarrow{d^{m+1}} \cdots$$
Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^\bullet(\mathbf{Ab})$. The *mapping cone* of the morphism u is the complex in $C^\bullet(\mathbf{Ab})$ given by
$$C^\bullet_u = Y^\bullet \oplus (X^\bullet[1]).$$

[7] Because of the rather dry and technical nature of the precise definitions, I have tried to give a more general idea about triangulated and derived categories; the following are a collection of concepts as opposed to a list of formal definitions and properties.

This gives rise to the *standard triangle* for a morphism u

$$T_u : X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{q} C_u^\bullet \xrightarrow{p} X^\bullet[1],$$

where q (resp. p) is the inclusion (resp. projection) morphisms. The standard triangle gives rise to long exact sequences in cohomology.

An important construction is that of a *homotopic* category[8]. This is an additive category $K^\bullet(\mathbf{Ab})$ with

- Obj $(K^\bullet(\mathbf{Ab})) = \text{Obj }(C^\bullet(\mathbf{Ab}))$,
- $\text{Hom}_{K^\bullet(\mathbf{Ab})}(X^\bullet, Y^\bullet) = \text{Hom}(X^\bullet, Y^\bullet)/\sim$ where $\sim$ is homotopy equivalence: $u \sim v \implies H^k(u) = H^k(v)$.

A family of triangles in $K^\bullet(\mathbf{Ab})$ is *distinguished* if they are isomorphic to a standard triangle for some morphism u.

In a homotopic category $K^\bullet(\mathbf{Ab})$, distinguished triangles satisfy a list of four properties referred to as TR1–TR4 in the literature. We will not repeat them here (see, for example, proposition 1.2.4 of [**22**]). It suffices to say that TR1–TR4 guarantees nice functorial properties of distinguished triangles, compatible with homotopy.

A *triangulated category* is an additive category $\mathcal{A}$ with a shift self-equivalence T, with $X[1] = TX$ and with a collection of distinguished triangles $\mathcal{T}$ that satisfy TR1–TR4 in the original definition due to Verdier. A *derived category* $D^\bullet(\mathbf{Ab})$ of an abelian category $\mathbf{Ab}$ is a triangulated category obtained from $K^\bullet(\mathbf{Ab})$ by localization with respect to the multiplicative system of quasi-isomorphisms in $K^\bullet(\mathbf{Ab})$.

1.5.2. Bloch–Ogus cohomology. The material for this section is taken from the seminal paper of Bloch–Ogus [**10**]. Bloch-Ogus cohomology is the mixed motives counterpart of Weil cohomology in the pure motives case, and as such, is a universal cohomology theory for schemes of a much general type.

Let $\mathbf{Sch}_k$ be the category of schemes of finite type over the field k. By $\mathbf{Sch}_k{}^*$ I mean the category with

- Objects: $Y \hookrightarrow X$ closed immersions in $X \in \text{Obj }(\mathbf{Sch}_k)$ and
- Morphisms: $\text{Hom}_{\mathbf{Sch}_k{}^*}((Y \hookrightarrow X), (Y' \hookrightarrow X'))$ being the following commutative cartesian squares:

$$\begin{array}{ccc} Y & \xrightarrow{\subset} & X \\ f_Y \downarrow & & \downarrow f_X \\ Y' & \xrightarrow{\subset} & X' \end{array}$$

Definition 1.56 (cohomology with supports). A *twisted cohomology theory with supports* is a sequence of contravariant functors

$$\begin{array}{rcl} H^i : \mathbf{Sch}_k{}^* & \longrightarrow & \mathbf{AbGp} \\ (Y \hookrightarrow X) & \mapsto & \bigoplus_i H^i_Y(X, n), \end{array}$$

satisfying the following:

[8] The reader is invited to compare this with the definition of a Grothendieck *group* in 1.2.

(1) For $Z \subseteq Y \subseteq X$, there is a long exact sequence

$$\cdots \longrightarrow H^i_Z(X,n) \longrightarrow H^i_Y(X,n) \longrightarrow H^i_{Y\setminus Z}(X\setminus Z,n) \longrightarrow H^{i+1}_Z(X,n) \longrightarrow \cdots .$$

(2) Let

$$\begin{array}{rcl} f:(Y\hookrightarrow X) & \longrightarrow & (Y'\hookrightarrow X'), \\ g:(Z\hookrightarrow X) & \longrightarrow & (Z'\hookrightarrow X') \end{array}$$

and $k:(Y\setminus Z\hookrightarrow X\setminus Z)\to(Y'\setminus Z'\hookrightarrow X'\setminus Z')$ be the induced arrow. Let $h:(Z\hookrightarrow X)\to(Z'\hookrightarrow X')$. Then the following diagram commutes:

$$\begin{array}{ccccccccc} \longrightarrow & H^i_Z(X,n) & \longrightarrow & H^i_Y(X,n) & \longrightarrow & H^i_{Y\setminus Z}(X\setminus Z,n) & \longrightarrow & H^{i+1}_Z(X,n) & \longrightarrow \\ & {\scriptstyle H^*(h)}\uparrow & & {\scriptstyle H^*(f)}\uparrow & & {\scriptstyle H^*(k)}\uparrow & & {\scriptstyle H^*(g)}\uparrow & \\ \longrightarrow & H^i_{Z'}(X',n) & \longrightarrow & H^i_{Y'}(X',n) & \longrightarrow & H^i_{Y'\setminus Z'}(X'\setminus Z',n) & \longrightarrow & H^{i+1}_{Z'}(X',n) & \longrightarrow \end{array}$$

(3) [Excision] Let $(Z\hookrightarrow X)\in$ Obj $(\mathbf{Sch}_k{}^*)$ and $(U\hookrightarrow X)$ be open in X containing Z. Then

$$H^i_Z(X,n)\xrightarrow{\sim} H^i_Z(U,n).$$

Dually we have

Definition 1.57. [Homology with supports] A *twisted homology with supports* is a sequence of covariant functors

$$H_*:\mathbf{Sch}_{k*}\longrightarrow \mathbf{AbGp}$$

where the category $\mathbf{Sch}_{k*}$ has objects of $\mathbf{Sch}_k$ and the morphisms are proper morphisms of $\mathbf{Sch}_k$ satisfying:

(1) H_* is a presheaf in étale topology. If $\alpha:X'\to X$ is an étale morphism there exists morphisms$\alpha^*:H_i(X,n)\to H_i(X',n)$.
(2) Let $\alpha:Y'\to Y$ and $\beta:X'\to X$ be étale. Let $f:X\to Y$ and $g:X'\to Y'$ be proper and consider the following cartesian square:

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha} & X' \end{array}$$

Then the following square commutes:

$$\begin{array}{ccc} H_i(X',n) & \xleftarrow{\beta^*} & H_i(X,n) \\ {\scriptstyle H_i(g,n)}\downarrow & & \downarrow{\scriptstyle H_i(f,n)} \\ H_i(Y',n) & \xleftarrow{\alpha^*} & H_i(Y,n) \end{array}$$

(3) Let $i:Y\hookrightarrow X$ be a closed immersion and $\alpha:X\setminus Y\hookrightarrow X$ be an open immersion. Then there exists a long exact sequence

$$\cdots\longrightarrow H_i(Y,n)\xrightarrow{i^*}H_i(X,n)\xrightarrow{\alpha^*}H_i(X\setminus Y,n)\longrightarrow H_{i-1}(Y,n)\longrightarrow\cdots .$$

(4) Let $f : X' \to X$ be proper and $Z = f(Z')$ for $Z \hookrightarrow X$. Let $\alpha : X' \setminus f^{-1}(Z) \hookrightarrow X' \setminus Z'$. Then the following square commutes:

$$\begin{array}{ccccccccc} \longrightarrow & H_i(Z',n) & \longrightarrow & H_i(X',n) & \longrightarrow & H_i(X'\setminus Z',n) & \longrightarrow & H_{i-1}(Z',n) & \longrightarrow \\ & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*\alpha^*} & & \downarrow{\scriptstyle f_*} & \\ \longrightarrow & H_i(Z,n) & \longrightarrow & H_i(X,n) & \longrightarrow & H_i(X\setminus Z,n) & \longrightarrow & H_{i-1}(Z,n) & \longrightarrow \end{array}$$

Definition 1.58 (Poincaré duality with supports). *A Poincaré duality theory for schemes of finite type with supports* is a twisted cohomology theory H^* with

(1) For all $Y \hookrightarrow X \in \text{Obj }(\mathbf{Sch}_k{}^*)$ there is a pairing

$$H_i(X,n) \times H^j_Y(X,n) \longrightarrow H_{i+j}(Y, m+n).$$

(2) If $Y \hookrightarrow X \in \text{Obj }(\mathbf{Sch}_k{}^*)$ and $(\beta \hookrightarrow \alpha) : (Y' \hookrightarrow X') \longrightarrow (Y \hookrightarrow X)$ an étale morphism in $\mathbf{Sch}_k{}^*$, then for $a \in H^j_Y(X,n)$ and $z \in H_i(X,m)$,

$$\alpha^*(a) \cap \alpha^*(Z) = \beta^*(a \cap z).$$

(3) [Projection] Let $f : (Y_1 \hookrightarrow X_1) \longrightarrow (Y_2 \hookrightarrow X_2)$ proper. Then for $a \in H^i_{Y_2}(X_2,n)$ and $z \in H_i(X_1,m)$,

$$H^i(f_X)(z) \cap a = H_i(f_Y)(z \cap H^i(f)(a)).$$

(4) [Fundamental class] Let $X \in \text{Obj }(\mathbf{Sch}_k)$ be irreducible and of dimension d. There exists a global section η_X of $H_{2d}(X,d)$ such if $\alpha : X' \longrightarrow X$ is étale, then $\alpha^*\eta_X = \eta_{X'}$.

(5) [Poincaré duality] Let $X \in \text{Obj }(\mathbf{Sch}_k)$ be smooth and of dimension d. Let $Y \hookrightarrow X$ be a closed immersion. Then

$$H^{2d-i}_Y(X, d-n) \xrightarrow{\cap \eta_X} H_i(Y,n)$$

is an isomorphism.

The main theorem of [**10**] is

Theorem 1.59. *Given a Poincaré duality theory, an étale morphism f_X and the commutative square*

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ {\scriptstyle f_Z}\downarrow & & \downarrow{\scriptstyle f_X} \\ Z & \xrightarrow{\subset} & X \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} H^i_{Z'}(X',n) & \xrightarrow{\cap \eta_{X'}} & H_{2d-i}(Z', d-n) \\ {\scriptstyle H^*(f)}\uparrow & & \uparrow{\scriptstyle f_Z^*} \\ H^i_Z(X,n) & \xrightarrow{\cap \eta_X} & H_{2d-i}(Z, d-n) \end{array}$$

1.5.3. Hodge structures. Let us quickly review the basic definitions of Hodge theory.

DEFINITION 1.60 (Pure Hodge structure). A pure Hodge structure of weight m on a finite dimensional vector space V is a decreasing filtration

$$\cdots \subset F^{p+1}V_{\mathbb{C}} \subset F^{p}V_{\mathbb{C}} \subset \cdots$$

with $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and satisfying the *Hodge decomposition*

$$V_{\mathbb{C}} = \oplus_{p+q=m} V^{p,q},$$

where $V^{p,q} = F^p \cap \overline{F^p V_{\mathbb{C}}}$ and denotes the conjugate filtration.

Hodge structures form a category **Hodge** with morphisms of vector spaces compatible with the filtration F making up the hom set. The category **Hodge** is a tensor category with the evident tensor product. Furthermore, by taking formal differences of Hodge structures $[H] - [H']$, we obtain the Grothendieck ring of Hodge structures $K(\textbf{Hodge})$.

Let $A \subset \mathbb{R}$. An A-mixed Hodge structure consists of the following data:

(1) An A-module of finite type V_A.
(2) An increasing filtration called the *weight filtration*

$$\cdots \subset W_n \subset W_{n+1} \subset \cdots$$

of $A \otimes \mathbb{Q}$-module $V_A \otimes \mathbb{Q}$.
(3) A decreasing filtration called the *Hodge filtration*

$$\cdots F^{p+1}V_{\mathbb{C}} \subset F^{p}V_{\mathbb{C}} \subset \cdots$$

where $V_{\mathbb{C}} \otimes V_A \otimes \mathbb{C}$.
(4) A *graded weight j factor* $\mathrm{gr}_j^W(V_A) := (W_j/W_{j-1}) \otimes \mathbb{C}$ with a pure Hodge structure induced by the filtration F and $\overline{F}$ on $V_{\mathbb{C}}$.

Mixed Hodge structures too form a category **MHodge** with morphisms $V_A \to V_{A'}$ A-module homomorphisms that are compatible with the Hodge and weight filtration. The following theorem is of great importance:

THEOREM 1.61 (Deligne). *The category **MHodge** is abelian with kernels and cokernels with induced filtration's.*

1.5.4. Mixed Tate motives over a number field. Mixed Tate motives can be defined as the triangulated subcategory of the triangulated category of mixed motives generated by the Tate objects $\mathbb{Q}(n)$. Over a number field, however, it is possible to obtain a nicer category in the following way.

Let k be a number field. Let $\mathbb{Q}(1)$ be the pure Tate motive. The category of mixed Tate motives over k, denoted as $\mathbf{MTMot}_k$ is constructed in the following way. Consider the simple objects $\mathbb{Q}(n)$ and assume that $\mathbb{Q}(a)$ and $\mathbb{Q}(b)$ are isomorphic for $a \neq b$ and that any simple object of $\mathbf{MTMot}_k$ is isomorphic to some $\mathbb{Q}(\cdot)$. Also consider the groups

$$\mathrm{Ext}^i_{\mathbf{MTMot}_k}(\mathbb{Q}(0), \mathbb{Q}(n))$$

and assume that they vanish for $i > 1$. An important result of Borel identifies these extensions:

THEOREM 1.62 (Borel). $\mathrm{Ext}^1_{\boldsymbol{MTMot}_k}(\mathbb{Q}(0), \mathbb{Q}(n)) \simeq K_{2n-1}(k) \otimes \mathbb{Q}$ *where* $K_{2n-1}(k)$ *is the Quillen* K*-theory of the field* k*.*

Now $K_{2n-1}(k) \otimes \mathbb{Q} = H_\bullet(GL(n,k))$, the linear group with entries in k, so the reader may think of the Quillen K-theory in this case in terms of this homological identification. We have the following fundamental theorem:

THEOREM 1.63 (Deligne–Goncharov). *The category* $\boldsymbol{MTMot}_k$ *for a number field* k *is a Tannakian category with objects* $\mathbb{Q}(n)$ *and the extensions described above. Furthermore, the Hodge realization functor* $\boldsymbol{MTMot}_k \to \boldsymbol{MHodge}$ *is exact and faithful.*

1.6. Motivic measures and zeta functions

The material for this section is based on [**21**] and [**34**].

1.6.1. Overview of basics. The *generalized Euler characteristic* χ associates to each object in $\mathbf{Var}_k$ an element in a fixed commutative ring R such that $\chi(X) = \chi(Y)$ for $X \simeq Y$ and $\chi(X) = \chi(Y) + \chi(X \setminus Y)$ for a closed subvariety $Y \subset X$. The product in R is given by the fibered product of varieties: $\chi(X \times Y) = \chi(X)\chi(Y)$. In fact the Euler characteristic is the canonical example of a generalized Euler characteristic.

Konstevich's original motivation behind inventing motivic measures was to prove the following theorem

THEOREM 1.64 (Kontsevich). *Let* X *and* Y *be two birationally equivalent Calabi-Yau manifolds. Then* X *and* Y *have the same Hodge numbers.*

These types of questions are of enormous importance for duality questions in string theory, namely in mirror symmetry. (There is a beautiful theory around this called the *geometric McKay correspondence* which I completely omit from the discussion.)

An example, following the proof of this theorem of Kontsevich, of a generalized Euler characteristic, comes from Hodge numbers of a complex manifold X. Recall the definition of mixed Hodge structures. For a Hodge structure V, we can define the class of V in $K(\mathbf{Hodge})$ in terms of the graded m-factors in the weight filtration: $[V] := \sum_m [\mathrm{gr}^W_m(V)]$. Define the *Hodge-Deligne polynomial* as

$$\chi_h(X) := \sum_i [H^i_c(X, \mathbb{Q})] \in K(\mathbf{Hodge})$$

where $H^i_c(X, \mathbb{Q})$ is the i-th cohomology of X with compact support. If $Y \subset X$ is locally closed, then the Hodge characteristic is compatible with the exact Gysin sequence:

$$\cdots \longrightarrow H^r_c(X \setminus Y, \mathbb{Q}) \longrightarrow H^r_c(X, \mathbb{Q}) \longrightarrow H^r_c(Y, \mathbb{Q}) \longrightarrow H^{r+1}_c(X \setminus Y, \mathbb{Q}) \longrightarrow \cdots.$$

That is

$$\chi_h(X) = \chi_h(Y) + \chi_h(X \setminus Y).$$

REMARK 1.65. For X the affine line $\mathbb{A}^1_k$, $H^r_c(\mathbb{A}^1_k, \mathbb{Q}) = 0$ for all $r \neq 2$ and $H^2_c(\mathbb{A}^1_k, \mathbb{Q})$ is one-dimensional of Hodge type $(1,1)$. So $\chi_h(\mathbb{A}^1_k)$ is invertible. This is the Lefschetz motive $\mathbb{L}$.

We have the following two ring homomorphisms

- [Hodge characteristic]

$$\chi_h : K(\mathbf{Hodge}) \to \mathbb{Z}[u, u^{-1}, v, v^{-1}]$$
$$H^i(X, \mathbb{Q}) \mapsto \sum_{p+q=i} \dim(H^{p,q}(X, \mathbb{Q}))u^p v^q$$

- [Weight characteristic]

$$\chi_{wt} : K(\mathbf{Hodge}) \to \mathbb{Z}[w, w^{-1}]$$
$$u, v \mapsto w.$$

Here $H^{p,q}(X, \mathbb{Q})$ is the (p, q)-th piece of $H^i_c(X, \mathbb{Q})$. Evaluating the weight characteristic at 1 gives us the topological Euler characteristic.

1.6.2. Equivariant Grothendieck ring. Denote by $\hat{\mu}$ the projective limit of the affine scheme of roots of unity μ_n:

$$\hat{\mu} := \varprojlim_n \mu_n = \varprojlim_n \mathrm{Spec} k[x]/(x^n - 1)$$

Let X be an S-variety. A *good μ_n-action on X* is a group action $\mu_n \times X \to X$ such that each orbit is contained in an affine subvariety of X. (A *good $\hat{\mu}$-action* factors through a good μ_n-action for some n.)

Definition 1.66 (equivariant Grothendieck ring). The equivariant Grothendieck ring of varieties $K^{\hat{\mu}}(\mathbf{Var}_{X_0})$ is a an abelian group generated by $[X, \hat{\mu}]_S$ for X an S-variety with good $\hat{\mu}$-action with the relations

(1) $[X, \hat{\mu}]_S = [Y, \hat{\mu}]_S$ for $X \simeq Y$ as S-varieties with good $\hat{\mu}$-action.
(2) $[X, \hat{\mu}]_S = [Y, \hat{\mu}]_S + [X \setminus Y, \hat{\mu}]_S$ if Y is closed in X and the $\hat{\mu}$-action on Y induced by $\hat{\mu}$-action on X.
(3) [monodromy] $[X \times V, \hat{\mu}]_S = [\mathbb{A}^n_k, \hat{\mu}]_S$ where V is the n-dimensional affine space with a good $\hat{\mu}$-action and $\mathbb{A}^n_k$ is the affine space with a trivial $\hat{\mu}$-action.

Definition 1.67 (equivariant Euler characteristic). The *equivariant Euler characteristic* is a ring homomorphism

$$\chi_{\mathrm{top}}(-, \alpha) : K^{\hat{\mu}}(\mathbf{Var}_{X_0})[\mathbb{L}^{-1}] \longrightarrow Z$$
$$(-) \mapsto \sum_{q \geq 0} (-1)^q \mathrm{dim} H^q(-, \mathbb{C})_\alpha$$

where $H^q(-, \mathbb{C})_\alpha \subset H^*(-, \mathbb{C})$ on which there is a good $\hat{\mu}$-action through multiplication by α.

1.6.3. Arc spaces. Let k be a field of characteristic zero and let X be a variety over k. For all natural numbers n, we define the *arc space* $\mathcal{L}_n(X)$ as an algebraic variety over k whose k-rational points are $K[t]/t^{n+1}$-rational points for some $k \subset K$. We set $\mathcal{L}(X) = \varprojlim_n \mathcal{L}_n(X)$. Note that $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$, the (Zariski) tangent space of X. We call the K-rational points of $\mathcal{L}_n(X)$ K-arcs of X or arcs for short. There are the structure morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X),$$
$$\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X).$$

The *origin* of an arc γ is $\pi_0(\gamma)$.

A "cleaner" (but equivalent) definition of the space of arcs is the following:

Definition 1.68 (Space of arcs). Let X be a variety over k (which, I remind the reader is a separated scheme of finite type over k.) Denote by $\mathcal{L}(X)$ the *scheme of germs of arcs on* X. It is a defined as a scheme over k such that, for any extension $k \subset K$ there is a natural bijection

$$\mathcal{L}(X)(K) \simeq \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec} k[[t]], K)$$

The scheme $\mathcal{L}(X) := \varprojlim_n \mathcal{L}_n(X)$ in the category of schemes $\mathcal{L}_n(X)$ representing the functor

$$R \mapsto \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec} R[t]/t^{n+1}, X)$$

defined on the category of K-algebras.

Not much is known about the space $\mathcal{L}(X)$. An important result is the following:

Proposition 1.69 (Kolchin). *Let X be an integral scheme. Then $\mathcal{L}(X)$ is irreducible.*

1.6.4. The Nash Problem. Arcs were first introduced by Nash in connection with singularities: let P be a singular point on X and set $\mathcal{L}_{\{P\}}(X) := \pi_0^{-1}(P)$ of arcs with origin P. He studied the space $\mathcal{N}_{\{P\}}(X) \subset \mathcal{L}_{\{P\}}(X)$ of arcs not contained in the singular locus Sing X. Let me briefly sketch Nash's beautiful idea, following Loeser's Trieste lectures [**35**].

Let $Y \xrightarrow{\rho} X$ be a resolution of singularities of a scheme X. Recall that this means: Y is smooth, ρ is proper and ρ induces an isomorphism $Y \setminus \rho^{-1}(\mathrm{Sing}\ X) \simeq X \setminus \mathrm{Sing}\ X$. We say that the resolution of singularities is *divisorial* if the locus E where ρ is *not* a local isomorphism (called the *exceptional set*) is a divisor in Y. In such cases, we call the locus an *exceptional divisor*. Let $Y' \xrightarrow{\rho'} X$ be another proper birational morphism (and assume that X is normal) and let p be the generic point of E. We say that E *appears in* ρ' if ${\rho'}^{-1} \circ \rho$ is a local isomorphism at p. The exceptional divisor E is called an *essential component* of a resolution of X if E appears in every divisorial resolution of X. Denote the set of essential components as $\mathcal{C}_{\{P\}}(X)$.

Theorem 1.70 (Nash). *The mapping $\mathcal{C}_{\{P\}}(X) \xrightarrow{\nu} \mathcal{N}_{\{P\}}(X)$ is injective.*

Basically, what this theorem tells is that every irreducible component of $\mathcal{L}(X)$ through a singularity P corresponds to an exceptional divisor that occurs on every resolution. The *Nash Problem* asks

Question 1.71. For what X is the map ν a bijection?

An explicit nonexample was provided by Ishii-Kollár when X is a 4-dimensional hypersurface singularity $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{A}^5_k | x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0\}$ which has 1 irreducible family of arcs but 2 essential components. (The base k is obviously not allowed to be of characteristic 2 or 3.)

1.6.5. Motivic zeta functions. Let t be a *fixed* coordinate on $\mathbb{A}^1_k$ and let $n \geq 1$ be an integer. A morphism $f : X \to \mathbb{A}^1_k$ induces a morphism $f_n : \mathcal{L}_n(X) \to \mathcal{L}_n(\mathbb{A}^1_k)$. Any $\alpha \in \mathcal{L}(\mathbb{A}^1_k)$ (resp. $\alpha \in \mathcal{L}_n(\mathbb{A}^1_k)$) gives a power series $\alpha(t) \in K[[t]]$ (resp. $\alpha(t) \in K[[t]]/t^{n+1}$) by definition. Define a map $\mathrm{ord}_t : \mathcal{L}(\mathbb{A}^1_k) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as $\mathrm{ord}_t(\alpha) := \max_{t^e | \alpha(t)}\{e\}$.

Definition 1.72. $\mathcal{X}_n := \{\phi \in \mathcal{L}_n(X) | \mathrm{ord}_t f_n(\phi) = n\}$.

The variety $\mathcal{X}_n$ is a locally closed subvariety of $\mathcal{L}_n(X)$. Let X_0 be a variety obtained by setting $f = 0$. The variety $\mathcal{X}_n$ is an X_0-variety through the structure maps π_0^n. Define the morphism

$$\overline{f}_n : \mathcal{X}_n \to \mathbb{G}_m \text{ given by } \phi \mapsto \text{coefficient of } t_n \text{ in } f_n(\phi)$$

and set $\mathcal{X}_{n,1} := \overline{f}_n^{-1}(1)$. The following characterization of $\mathcal{X}_n$ allows us to pass on to étale settings "without reductions at bad primes".

Proposition 1.73. *As $\mathbb{G}_m \times X_0$-variety, $\mathcal{X}_n$ is a quotient of $\mathcal{X}_{n,1} \times \mathbb{G}_m$ by the μ_n-action $a(\phi, b) = (a\phi, a^{-1}b)$.*

Recall the definition of the equivariant Grothendieck ring $K^{\hat{\mu}}(\mathbf{Var}_{X_0})$ of section 1.6.2.

Definition 1.74 (Denef-Loeser, Looijenga). The *motivic zeta function* of a morphism $f : X \to \mathbb{A}^1_k$ over $K^{\hat{\mu}}(\mathbf{Var}_{X_0})[\mathbb{L}^{-1}]$ is defined by the sum

$$Z(T) = \sum_{n \geq 1} [\mathcal{X}_{n+1}, \hat{\mu}]_{X_0} \mathbb{L}^{-nd} T^n$$

where $[-, \hat{\mu}]_{X_0}$ is a class in $K^{\hat{\mu}}(\mathbf{Var}_{X_0})$ and $d = \dim \mathcal{X}_{n+1}$.

Theorem 1.75 (Denef-Loeser). *The motivic zeta function $Z(T)$ in definition 1.74 is a rational function.*

1.7. Feynman integrals and periods

1.7.1. Feynman's trick and Schwinger parameters. We begin by describing a simple generalization of the well known "Feynman trick",

$$\frac{1}{ab} = \int_0^1 \frac{1}{(xa + (1 - x)b)^2} dx,$$

which will be useful in the following. The results recalled here are well known in the physics literature (see *e.g.* [**90**] §8 and §18), but we give a brief and self contained treatment here for the reader's convenience. A similar derivation from a more algebro-geometric viewpoint can be found in [**92**].

LEMMA 1.76. *Let Σ_n denote the n-dimensional simplex*

$$\Sigma_n = \{(t_1, \dots, t_n) \in (\mathbb{R}_+^*)^n \mid \sum_{i=1}^n t_i \le 1\}. \tag{1.7.1}$$

Let $dv_{\Sigma_n} = dt_1 \cdots dt_{n-1}$ be the volume form on Σ_n induced by the standard Euclidean metric in $\mathbb{R}^n$. Then, for given nonzero parameters q_i, for $i = 1, \dots, n$, the following identity holds:

$$\frac{1}{q_1 \dots q_n} = (n-1)! \int_{\Sigma_{n-1}} \frac{1}{(t_1 q_1 + \dots + t_n q_n)^n} dv_{\Sigma_n}, \tag{1.7.2}$$

where $t_n = 1 - \sum_{i=1}^{n-1} t_i$.

PROOF. The following identity holds:

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{1}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \dots + s_n q_n)}\, s_1^{k_1 - 1} \cdots s_n^{k_n - 1}\, ds_1 \cdots ds_n. \tag{1.7.3}$$

The s_i are usually called Schwinger parameters in the physics literature. We then perform a change of variables, by setting $S = \sum_{i=1}^n s_i$ and $s_i = St_i$, with $t_i \in [0,1]$ with $\sum_{i=1}^n t_i = 1$. Thus, we rewrite (1.7.3) in the form

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{\Gamma(k_1 + \dots + k_n)}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_0^1 \cdots \int_0^1 \frac{t_1^{k_1 - 1} \cdots t_n^{k_n - 1}\, \delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \dots + t_n q_n)^{k_1 + \dots + k_n}}\, dt_1 \cdots dt_n. \tag{1.7.4}$$

The result (1.7.2) then follows as a particular case of this more general identity, with $k_i = 1$ for $i = 1, \dots, n$ and $\Gamma(n) = (n-1)!$. □

One can also give an inductive proof of (1.7.2) by Stokes theorem, which avoids introducing any transcendental functions, but the argument we recalled here is standard and it suffices for our purposes.

The Feynman trick is then related to the graph polynomial Ψ_Γ in the following way (see again [**90**], §18 and [**102**]). Suppose given a graph Γ. Let $n = \#E(\Gamma)$ be the number of edges of Γ and let $\ell = b_1(\Gamma)$ be the number of loops, *i.e.* the rank of $H^1(\Gamma, \mathbb{Z})$. Suppose chosen a set of generators $\{l_1, \dots, l_\ell\}$ of $H^1(\Gamma, \mathbb{Z})$. We then define the $n \times \ell$-matrix η_{ik} as

$$\eta_{ik} = \begin{cases} +1 & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation} \\ -1 & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation} \\ 0 & \text{otherwise.} \end{cases} \tag{1.7.5}$$

Also let M_Γ be the $\ell \times \ell$ real symmetric matrix

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^n t_i \eta_{ik} \eta_{ir}, \tag{1.7.6}$$

for $t = (t_0, \dots, t_{n-1}) \in \Sigma_n$ and $t_n = 1 - \sum_i t_i$. Let s_k, $k = 1, \dots, \ell$ be real variables $s_k \in \mathbb{R}^D$ assigned to the chosen basis of the homology $H^1(\Gamma, \mathbb{Z})$. Also let p_i, for $i = 1, \dots, n$ be real variables $p_i \in \mathbb{R}^D$ associated to the edges of Γ. Let $q_i(p)$ denote the quadratic form

$$q_i(p) = p_i^2 - m_i^2, \tag{1.7.7}$$

for fixed parameters $m_i > 0$. These correspond to the Feynman propagators

$$\frac{1}{q_i} = \frac{1}{p_i^2 - m_i^2} \tag{1.7.8}$$

for a scalar field theory, associated by the Feynman rules to the edges of the graph. One can make a change of variables

$$p_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} s_k, \quad \text{with the constraint} \quad \sum_{i=0}^{n} t_i u_i \eta_{ik} = 0.$$

Then we have the following result.

Lemma 1.77. *The following identity holds*

$$\int \frac{1}{(\sum_{i=0}^n t_i q_i)^n} \, d^D s_1 \cdots d^D s_\ell = C_{\ell,n} \, \det(M_\Gamma(t))^{-D/2} (\sum_{i=0}^{n} t_i (u_i^2 - m_i^2))^{-n+D\ell/2}. \tag{1.7.9}$$

Proof. After the change of variables, the left hand side reads

$$\int \frac{d^D s_1 \cdots d^D s_\ell}{(\sum_{i=0}^n t_i(u_i^2 - m_i^2) + \sum_{kr} (M_\Gamma)_{kr} s_k s_r)^n} \,.$$

The integral can then be reduced by a change of variables that diagonalizes the matrix M_Γ to an integral of the form

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{(a - \sum_k \lambda_k x_k^2)^n} = C_{\ell,n} \, a^{-n+D\ell/2} \prod_{k=1}^{\ell} \lambda_k^{-D/2},$$

with

$$C_{\ell,n} = \int \frac{d^D x_1 \cdots d^D x_\ell}{(1 - \sum_k x_k^2)^n}.$$

□

This is the basis for the well known formula that relates the computation of Feynman integrals to periods, used in [**92**]. In fact, we have the following.

Corr In the case of graphs where the number of edges and the number of loops are related by $n = D\ell/2$, the Feynman integral is computed by

$$\int \frac{d^D s_1 \cdots d^D s_\ell}{q_0 \cdots q_n} = C_{\ell,n} \int_{\Sigma_n} \frac{dt_0 \cdots dt_{n-1}}{\Psi_\Gamma(t_0, \ldots, t_n)^{D/2}}, \tag{1.7.10}$$

where $t_n = 1 - \sum_{i=0}^{n-1} t_i$ and

$$\Psi_\Gamma(t) = \det(M_\Gamma(t)). \tag{1.7.11}$$

Proof. Notice that, in the case of graphs with $n = D\ell/2$, the integration (1.7.9) reduces to

$$\int \frac{d^D s_1 \cdots d^D s_\ell}{(\sum_{i=0}^n t_i q_i)^n} = C_{\ell,n} \, \det(M_\Gamma(t))^{-D/2}. \tag{1.7.12}$$

□

Write $t = (t_1, \ldots, t_n)$ and $\det(M_\Gamma(t))$ as $\Psi_\Gamma(t)$. Then we have the following theorem:

THEOREM 1.78 (cf. chapter 6 of [**114**]). *Let Γ be a graph and S range over all $S \subset E_{int}(\Gamma)$ of ℓ edges such that removal of every edge in S leaves a connected graph. Then*

$$\Psi_\Gamma(t_1, \ldots, t_n) = \sum_S \prod_{e \in S} t_e. \tag{1.7.13}$$

PROOF. See Proposition 3.1.6 of [**99**]. □

There is an equivalent formulation of $\Psi_\Gamma(t)$ as

$$\Psi_\Gamma(t) = \sum_T \prod_{e \in T} t_e \tag{1.7.14}$$

or equivalently (in a form we would use)

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e. \tag{1.7.15}$$

Here T is a spanning tree of Γ, e an edge in T and t_e a formal variable associated to it. (For a proof of these facts, see [**102**], section 1.3.)

1.7.2. Properties of graph polynomials. Graph polynomials satisfy several properties. Among them, one notes (proposition 2.6 of [**92**]) that

(1) Ψ_Γ are sums of monomials with coefficient $+1$.
(2) No variables of degree > 1 appear in any monomials in Ψ_Γ.
(3) Ψ_Γ is a homogenous polynomial of degree = number of cycles of Γ.

Note that we are using the language of graph theory and call a *cycle* of Γ (a generator of $H_1(\Gamma, \mathbb{Z})$) what many physicists call a loop. In this language, the number of loops of Γ will be $h_1(\Gamma) :=$ rank $H_1(\Gamma, \mathbb{Z})$.

1.7.3. Graph hypersurfaces. The Kirchhoff polynomials define hypersurfaces

$$X_\Gamma = \{t = (t_e) \in \mathbb{P}^{\#E(\Gamma)-1} \mid \Psi_\Gamma(t) = 0\}. \tag{1.7.16}$$

These are typically singular hypersurfaces.

DEFINITION 1.79 (Feynman period). With the notation above, the Feynman period is defined in the log-divergent case as

$$U(\Gamma) = \int_{\Sigma_n} \frac{\Omega}{\Psi_\Gamma{}^{\frac{D}{2}}} \tag{1.7.17}$$

with $\Omega = \sum_{i=0}^n (-1)^{i+1} t_i dt_1 \wedge \ldots \wedge \widehat{dt_j} \wedge \ldots \wedge dt_n$ the volume form in $\mathbb{P}^{n-1}(\mathbb{R})$

REMARK 1.80 ([**92**], [**8**]). The basic geometric input data is the pair

$$(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Delta)$$

with $\Delta = \{\prod_{i=1}^n t_i = 0\}$. With this, the boundary of the chain Σ_n is supported on Δ. Renormalization in their point of view is necessary whenever $\frac{\Omega}{\Psi_\Gamma{}^{\frac{D}{2}}}$ acquires poles along the expectional divisors upon blowing up the faces of Δ.

In the case of the log divergent graphs as considered in [**92**] with number of loops = 2 (number of edges), the motive involved in the evaluation of the Feynman integral as a period is of the form

$$H^{2n-1}(P \smallsetminus Y_\Gamma, \Sigma \smallsetminus (\Sigma \cap Y_\Gamma)), \tag{1.7.18}$$

where n is the number of loops, $P \to \mathbb{P}^{2n-1}$ is a blowup along linear spaces, Y_Γ is the strict transform of X_Γ and Σ is the total inverse image of the coordinate simplex of $\mathbb{P}^{2n-1}$. In case of the wheel-with-n-spokes graph, it is shown that (1.7.18) viewed as a Hodge-Tate structure is $\mathbb{Q}(-2)$ and as an algebraic de-Rham realization is generated by the integrand of 1.7.17. Doryn in his PhD thesis used the methods of [**92**] to obtain a similar result for the family of zig-zag graphs.

However, there is no reason to believe that *all* motives obtained from Feynman graph would be mixed Tate. In fact, a very influential paper of Belkale-Brosnan (by way of disproving a point-counting conjecture of Kontsevich) show that hypersurfaces from Feyman graphs generate the entire Grothendieck ring of $\mathbb{Q}$-quasiprojective varieties. This imply the existence of motives associated to Feynman graphs that are not mixed Tate which, in turn, imply the existence of Feynman periods that are not (multiple) zeta values, following the theory of Deligne–Goncharov.

CHAPTER 2

Supergeometry of Feynman graphs

This Chapter is the result of my joint work with Matilde Marcolli.

2.1. Introduction

The fact, due to Belkale–Brosnan, that the graph hypersurfaces generate the Grothendieck ring of varieties means that the computation of the Feynman integral in terms of a period on the complement of a graph hypersurface in a projective space gives a general procedure to construct a large class of interesting varieties with associated periods. Our purpose here is to show that this general procedure can be adapted to produce a large class of interesting supermanifolds with associated periods.

In the setting of [**92**] and [**91**] one is assuming, from the physical viewpoint, that all edges of the graph are of the same nature, as would be the case in a scalar field theory with Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi). \tag{2.1.1}$$

However, in more general theories, one has graphs that are constructed out of different types of edges, which correspond to different propagators in the corresponding Feynman rules. We consider the case of theories with fermions, where graphs have both *fermionic* and *bosonic* legs. From the mathematical point of view, it is natural to replace the usual construction of the graph hypersurface by a different construction which assigns to the edges either ordinary variables (bosonic) or Grassmann variables (fermionic). This procedure yields a natural way to construct a family of *supermanifolds* associated to this type of Feynman graphs.

We give a computation of the Feynman integral in terms of a bosonic and a fermionic integration, so that the integral is computed as a period on a supermanifold that is the complement of a divisor in a superprojective space, defined by the set of zeros and poles of the Berezinian of a matrix $\mathcal{M}(t)$ associated to a graph Γ and a choice of a basis B for $H_1(\Gamma)$. We refer to the divisor defined by this Berezinian as the graph supermanifold $\mathcal{X}_{(\Gamma,B)}$.

As in the case of the ordinary graph hypersurfaces, we are interested in understanding their motiving nature first by looking at their classes in the Grothendieck ring of varieties. To this purpose, we introduce a Grothendieck ring $K_0(\mathcal{SV}_{\mathbb{C}})$ of supermanifold and we prove that it is a polynomial ring $K_0(\mathcal{V}_{\mathbb{C}})[T]$ over the Grothendieck ring of ordinary varieties. We then use this result to prove that the classes of the graph supermanifolds $\mathcal{X}_{(\Gamma,B)}$ generate the subring $K_0(\mathcal{V}_{\mathbb{C}})[T^2]$, where the degree two appears due to a fermion doubling used in the computation of the Feynman integral.

In a different perspective, an interest in supermanifolds and their periods has recently surfaced in the context of mirror symmetry, see [**107**], [**86**], [**96**]. We do not know, at present, whether the classes of supermanifolds considered here and their periods may be of any relevance to that context. We mention some points of contact in §2.4 below.

As the referee to the published version of this Chapter pointed out to us, a theory of parametric Feynman integrals for scalar supersymmetric theories was developed in [**101**]. The type of integrals we are considering here is slightly different from those of [**101**], hence we cannot apply directly the results of that paper. It would be interesting to see what class of graph supermanifolds can be obtained from the parametric integrals of [**101**].

2.2. Supermanifolds and motives

2.2.1. Supermanifolds. We recall here a few basic facts of supergeometry that we need in the following. The standard reference for the theory of supermanifolds is Manin's [**100**].

By a complex supermanifold one understands a datum $\mathcal{X} = (X, \mathcal{A})$ with the following properties: $\mathcal{A}$ is a sheaf of supercommutative rings on X; $(X, \mathcal{O}_X)$ is a complex manifold, where $\mathcal{O}_X = \mathcal{A}/\mathcal{N}$, with $\mathcal{N}$ the ideal of nilpotents in $\mathcal{A}$; the quotient $\mathcal{E} = \mathcal{N}/\mathcal{N}^2$ is locally free over $\mathcal{O}_X$ and $\mathcal{A}$ is locally isomorphic to the exterior algebra $\Lambda^\bullet_{\mathcal{O}_X}(\mathcal{E})$, where the grading is the $\mathbb{Z}_2$-grading by odd/even degrees. The supermanifold is split if the isomorphism $\mathcal{A} \cong \Lambda^\bullet_{\mathcal{O}_X}(\mathcal{E})$ is global.

EXAMPLE 2.1. Projective superspace. The complex projective superspace $\mathbb{P}^{n|m}$ is the supermanifold $(X, \mathcal{A})$ with $X = \mathbb{P}^n$ the usual complex projective space and

$$\mathcal{A} = \Lambda^\bullet(\mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{O}(-1)),$$

with the exterior powers $\Lambda^\bullet$ graded by odd/even degree. It is a split supermanifold.

A *morphism* $F : \mathcal{X}_1 \to \mathcal{X}_2$ of supermanifolds $\mathcal{X}_i = (X_i, \mathcal{A}_i)$, $i = 1, 2$, consists of a pair $F = (f, f^\#)$ of a morphism of the underlying complex manifolds $f : X_1 \to X_2$ together with a morphism $f^\# : \mathcal{A}_2 \to f_*\mathcal{A}_1$ of sheaves of supercommutative rings with the property that at each point $x \in X_1$ the induced morphism $f^\#_x : (\mathcal{A}_2)_{f(x)} \to (\mathcal{A}_1)_x$ satisfies $f^\#_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$, on the maximal ideals of germs of sections vanishing at the point (*cf.* [**100**], §4.1).

In particular, an *embedding* of complex supermanifolds is a morphism $F = (f, f^\#)$ as above, with the property that $f : X_1 \hookrightarrow X_2$ is an embedding and $f^\# : \mathcal{A}_2 \to f_*\mathcal{A}_1$ is surjective. As in ordinary geometry, we define the ideal sheaf of $\mathcal{X}_1$ to be the kernel

$$\mathcal{I}_{\mathcal{X}_1} := \mathrm{Ker}(f^\# : \mathcal{A}_2 \to f_*\mathcal{A}_1). \tag{2.2.1}$$

An equivalent characterization of an embedding of supermanifold is given as follows. If we denote by E_i, for $i = 1, 2$ the holomorphic vector bundles on X_i such that $\mathcal{O}(E_i) = \mathcal{E}_i = \mathcal{N}_i/\mathcal{N}_i^2$, with the notation as above, then an embedding $F : \mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ is an embedding $f : X_1 \hookrightarrow X_2$ such that the induced morphism of vector bundles $f^*: E_2 \to E_1$ is surjective (*cf.* [**98**]). Thus, we say that $\mathcal{Y} = (Y, \mathcal{B})$ is a closed sub-supermanifold of $\mathcal{X} = (X, \mathcal{A})$ when there exists a closed embedding $Y \subset X$ and the pullback of $E_{\mathcal{A}}$ under this embedding surjects to $E_{\mathcal{B}}$.

An open submanifold $\mathcal{U} = (U, \mathcal{B}) \hookrightarrow \mathcal{X} = (X, \mathcal{A})$ is given by an open embedding $U \hookrightarrow X$ of the underlying complex manifolds and an isomorphism of sheaves $\mathcal{A}|_U \cong \mathcal{B}$. When $\mathcal{Y} \subset \mathcal{X}$ is a closed embedding and $U = X \setminus Y$, the ideal sheaf of $\mathcal{Y}$ satisfies $\mathcal{I}_{\mathcal{Y}}|_U = \mathcal{A}|_U$.

A subvariety in superprojective space is a supermanifold

$$\mathcal{X} = (X \subset \mathbb{P}^n, (\Lambda^\bullet(\mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{O}(-1))/\mathcal{I})|_X), \tag{2.2.2}$$

where $\mathcal{I} = \mathcal{I}_{\mathcal{X}}$ is an ideal generated by finitely many homogeneous polynomials of given $\mathbb{Z}/2$-parity. In other words, if we denote by $(x_0, \ldots, x_n, \theta_1, \ldots, \theta_m)$ the bosonic and fermionic coordinates of $\mathbb{P}^{n|m}$, then a projective subvariety can be obtained by assigning a number of equations of the form

$$\Psi^{ev/odd}(x_0, \ldots, x_n, \theta_1, \ldots, \theta_m) = \sum_{i_1 < \cdots < i_k} P_{i_1, \ldots, i_k}(x_0, \ldots, x_n)\theta_{i_1} \cdots \theta_{i_k} = 0, \tag{2.2.3}$$

where the $P_{i_1, \ldots, i_k}(x_0, \ldots, x_n)$ are homogeneous polynomials in the bosonic variables.

Notice that there are strong constraints in supergeometry on realizing supermanifolds as submanifolds of superprojective space. For instance, Penkov and Skornyakov [**103**] showed that super Grassmannians in general do not embed in superprojective space, *cf.* [**100**]. The result of LeBrun, Poon, and Wells [**98**] shows that a supermanifold $\mathcal{X} = (X, \mathcal{A})$ with compact X can be embedded in some superprojective space $\mathbb{P}^{n|m}$ if and only if it has a positive rank-one sheaf of $\mathcal{A}$-modules.

Notice that, in the above, we have been working with complex projective superspace and complex subvarieties. However, it is possible to consider supergeometry in an arithmetic context, as shown in [**106**], so that it makes sense to investigate extensions of motivic notions to the supergeometry setting. In the present paper we limit our investigation of motivic aspects of supermanifolds to the analysis of their classes in a suitable Grothendieck ring.

2.2.2. A Grothendieck group. We begin by discussing the Grothendieck group of varieties in the supergeometry context and its relation to the Grothendieck group of ordinary varieties. The latter case of the Grothendieck ring in case of ordinary varieties is discussed in Chapter 1, section 1.1.

We first recall the following notation from [**95**] §II.2.3. Given a locally closed subset $Y \subset X$ and a sheaf $\mathcal{A}$ on X, there exists a unique sheaf $\mathcal{A}_Y$ with the property that

$$\mathcal{A}_Y|_Y = \mathcal{A}|_Y \quad \text{and} \quad \mathcal{A}_Y|_{X \setminus Y} = 0. \tag{2.2.4}$$

In the case where Y is closed, this satisfies $\mathcal{A}_Y = i_*(\mathcal{A}|_Y)$ where $i: Y \hookrightarrow X$ is the inclusion, and when Y is open it satisfies $\mathcal{A}_Y = \mathrm{Ker}(\mathcal{A} \to i_*(\mathcal{A}|_{X \setminus Y}))$.

Definition 2.2. Let $\mathcal{SV}_{\mathbb{C}}$ be the category of complex supermanifolds with morphisms defined as above. Let $K_0(\mathcal{SV}_{\mathbb{C}})$ denote the free abelian group generated by the isomorphism classes of objects $\mathcal{X} \in \mathcal{SV}_{\mathbb{C}}$ subject to the following relations. Let $F : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed embedding of supermanifolds. Then

$$[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} \smallsetminus \mathcal{Y}], \tag{2.2.5}$$

where $\mathcal{X} \smallsetminus \mathcal{Y}$ is the supermanifold

$$\mathcal{X} \smallsetminus \mathcal{Y} = (X \smallsetminus Y, \mathcal{A}_X|_{X \smallsetminus Y}). \tag{2.2.6}$$

In particular, in the case where $\mathcal{A} = \mathcal{O}_X$ is the structure sheaf of X, the relation (2.2.5) reduces to the usual relation

$$[X] = [Y] + [X \smallsetminus Y]. \tag{2.2.7}$$

in the Grothendieck group of ordinary varieties, for a closed embedding $Y \subset X$.

Lemma 2.3. *All supermanifolds decompose in $K_0(\mathcal{SV}_{\mathbb{C}})$ as a finite combination of split supermanifolds, and in fact of supermanifolds where the vector bundle E with $\mathcal{O}(E) = \mathcal{E} = \mathcal{N}/\mathcal{N}^2$ is trivial.*

Proof. This is a consequence of the dévissage of coherent sheaves. Namely, for any coherent sheaf $\mathcal{A}$ over a Noetherian reduced irreducible scheme there exists a dense open set U such that such that $\mathcal{A}|_U$ is free. The relation (2.2.5) then ensures that, given a pair $\mathcal{X} = (X, \mathcal{A})$ and the sequence of sheaves

$$0 \to i_!(\mathcal{A}|_U) \to \mathcal{A} \to j_*(\mathcal{A}|_Y) \to 0,$$

associated to the open embedding $U \subset X$ with complement $Y = X \smallsetminus U$, the class $[X, \mathcal{A}]$ satisfies

$$[X, \mathcal{A}] = [U, \mathcal{A}_U|_U] + [Y, \mathcal{A}_Y|_Y].$$

The sheaf $\mathcal{A}_Y$ on X, which has support Y, has a chain of subsheaves $\mathcal{A}_Y \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_k = 0$ such that each quotient $\mathcal{A}_i/\mathcal{A}_{i+1}$ is coherent on Y. Thus, one can find a stratification where on each open stratum the supermanifold is split and with trivial vector bundle. The supermanifold $\mathcal{X} = (X, \mathcal{A})$ decomposes as a sum of the corresponding classes in the Grothendieck group. □

The fact that the vector bundle that constitutes the fermionic part of a supermanifold is trivial when seen in the Grothendieck group is the analog for supermanifolds of the fact that any projective bundle is equivalent to a product in the Grothendieck group of ordinary varieties.

It follows from Lemma 2.3 above that the product makes $K_0(\mathcal{SV}_{\mathbb{C}})$ into a ring with

$$[\mathcal{X}][\mathcal{Y}] = [\mathcal{X} \times \mathcal{Y}].$$

In fact, we have the following more precise description of $K_0(\mathcal{SV}_{\mathbb{C}})$ in terms of the Grothendieck ring of ordinary varieties.

Corollary 2.4. The Grothendieck ring $K_0(\mathcal{SV}_{\mathbb{C}})$ of supervarieties is a polynomial ring over the Grothendieck ring of ordinary varieties of the form

$$K_0(\mathcal{SV}_{\mathbb{C}}) = K_0(\mathcal{V}_{\mathbb{C}})[T], \tag{2.2.8}$$

where $T = [\mathbb{A}^{0|1}]$ is the class of the affine superspace of dimension $(0, 1)$.

It then follows that the relation in theorem [**40**] between the Grothendieck ring and the semigroup ring of stable birational equivalence classes extends to this context.

Notice that, in the supermanifold case, there are now two different types of Lefschetz motives, namely the bosonic one $\mathbb{L}_b = [\mathbb{A}^{1|0}]$ and the fermionic one $\mathbb{L}_f = [\mathbb{A}^{0|1}]$. By analogy to what happens in motivic integration and in the theory of motives, we may want to localize at the Lefschetz motives, *i.e.* invert both $\mathbb{L}_b$ and $\mathbb{L}_f$. That is, according to Corollary 2.4, we consider the field of fractions of $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}_b^{-1}] = \mathcal{S}^{-1}K_0(\mathcal{V}_{\mathbb{C}})$, with respect to the multiplicative semigroup $\mathcal{S} = \{1, \mathbb{L}_b, \mathbb{L}_b^2, \ldots\}$ and then the ring of Laurent polynomials

$$\mathcal{S}^{-1}K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}_f, \mathbb{L}_f^{-1}] = K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}_b^{-1}, \mathbb{L}_f, \mathbb{L}_f^{-1}]. \tag{2.2.9}$$

This suggests extensions of motivic integration to the context of supermanifolds, but we will not pursue this line of thought further in the present paper.

There is also a natural extension to supermanifolds of the usual notion of birational equivalence. We say that two supermanifolds $\mathcal{X} = (X, \mathcal{A})$ and $(Y, \mathcal{B})$ are birational if there exist open dense embeddings of supermanifolds $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$ and an isomorphism $\mathcal{U} \cong \mathcal{V}$. Similarly, one can give a notion analogous to that of stable birational equivalence by requiring that there are superprojective spaces $\mathbb{P}^{n|m}$ and $\mathbb{P}^{r|s}$ such that $\mathcal{X} \times \mathbb{P}^{n|m}$ and $\mathcal{Y} \times \mathbb{P}^{r|s}$ are birational. One then finds the following. We denote by $\mathbb{Z}[SSB]$ the semigroup ring of stable birational equivalence classes of supermanifolds.

COROLLARY 2.5. There is a surjective ring homomorphism $K_0(\mathcal{S}\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[SSB]$, which induces an isomorphism

$$K_0(\mathcal{S}\mathcal{V}_{\mathbb{C}})/I \cong \mathbb{Z}[SSB], \tag{2.2.10}$$

where I is the ideal generated by the classes $[\mathbb{A}^{1|0}]$ and $[\mathbb{A}^{0|1}]$.

The formal inverses of $\mathbb{L}_f$ and $\mathbb{L}_b$ correspond to two types of Tate objects for motives of supermanifold, respectively fermionic and bosonic Tate motives. We see from Corollary 2.4 and (2.2.9) that the fermionic part of the supermanifolds only contribution to the class in the Grothendieck ring is always of this fermonic Tate type, while only the bosonic part can provide non-Tate contributions.

2.2.3. Integration on supermanifolds. The analog of the determinant in supergeometry is given by the Berezinian. This is defined in the following way. Suppose given a matrix $\mathcal{M}$ of the form

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where the M_{11} and M_{22} are square matrices with entries of order zero and the M_{12} and M_{21} have elements of order one. Then the Berezinian of $\mathcal{M}$ is the expression

$$\mathrm{Ber}(\mathcal{M}) := \frac{\det(M_{11} - M_{12}M_{22}^{-1}M_{21})}{\det(M_{22})}. \tag{2.2.11}$$

It satisfies $\mathrm{Ber}(\mathcal{M}\mathcal{N}) = \mathrm{Ber}(\mathcal{M})\mathrm{Ber}(\mathcal{N})$.

It is shown in [**89**] that Grassmann integration satisfies a change of variable formula where the Jacobian of the coordinate change is given by the Berezinian

$\mathrm{Ber}(J)$ with J the matrix $J = \frac{\partial X_\alpha}{\partial Y_\beta}$ with $X_\alpha = (x_i, \xi_r)$ and $Y_\beta = (y_j, \eta_s)$. We explain in §2.3 below how to use this to replace expressions of the form (**??**) for Feynman integrals, with similar expressions involving a Berezinian and a Grassmann integration over a supermanifold.

2.2.4. Divisors. There is a well developed theory of divisors on supermanifolds, originating from [**105**]. A Cartier divisor on a supermanifold $\mathcal{X}$ of dimension $(n|m)$ is defined by a collection of *even* meromorphic functions ϕ_i defined on an open covering $\mathcal{U}_i \hookrightarrow \mathcal{X}$, with $\phi_i \phi_j^{-1}$ a holomorphic function on $\mathcal{U}_i \cap \mathcal{U}_j$ nowhere vanishing on the underlying $U_i \cap U_j$. Classes of divisors correspond to equivalence classes of line bundles and can be described in terms of integer linear combinations of $(n-1|m)$-dimensional subvarieties $\mathcal{Y} \subset \mathcal{X}$.

2.3. Supermanifolds from graphs

We start with a modified version of the construction in section of the previous Chapter, where we deal with graphs that have both bosonic and fermionic legs, and we maintain the distinction between these two types at all stages by assigning to them different sets of ordinary and Grassmann variables. Strictly from the physicists point of view this is an unnecessary complication, because the formulae we recalled in this section adapt to compute Feynman integrals also in theories with fermionic fields, but from the mathematical viewpoint this procedure will provide us with a natural way of constructing an interesting class of supermanifolds with associated periods.

2.3.1. The case of Grassmann variables. Consider now the case of Feynman propagators and Feynman diagrams that come from theories with both bosonic and fermionic fields. This means that, in addition to terms of the form (2.1.1), the Lagrangian also contains fermion interaction terms. The form of such terms is severely constrained (see *e.g.* [**104**], §5.3): for instance, in dimension $D = 4$ renormalizable interaction terms can only involve at most two fermion and one boson field.

The perturbative expansion for such theories corresponsingly involve graphs Γ with two different types of edges: fermionic and bosonic edges. The Feynman rules assign to each bosonic edge a propagator of the form (1.7.8) and to fermionic edges a propagator

$$i\frac{\mathrm{p}+m}{p^2-m^2} = \frac{i}{\mathrm{p}-m}. \tag{2.3.1}$$

Notice that in physically significant theories one would have $i(\not{p}-m)^{-1}$ with $\not{p} = p^\mu \gamma_\mu$, but for simplicity we work here with propagators of the form (2.3.1), without tensor indices.

In the following we use the notation

$$q(p) = p^2 - m^2, \quad \mathrm{q}(p) = i(\mathrm{p}+m) \tag{2.3.2}$$

for the quadratic and linear forms that appear in the propagators (1.7.8) and (2.3.1). In the following, again just to simplify notation, we also drop the mass terms in the propagator (*i.e.* we set $m = 0$) and ignore the resulting infrared divergence problem. The reader can easily reinstate the masses whenever needed.

Thus, the terms of the form $(q_1 \cdots q_n)^{-1}$, which we encountered in the purely bosonic case, are now replaced by terms of the form

$$\frac{\mathrm{q}_1 \cdots \mathrm{q}_f}{q_1 \cdots q_n}, \tag{2.3.3}$$

where $n = \#E(\Gamma)$ is the total number of edges in the graph and $f = \#E_f(\Gamma)$ is the number of fermionic edges.

We first prove an analog of Lemma 1.76, where we now introduce an extra set of Grassmann variables associated to the fermionic edges. The derivation we present suffers from a kind of "fermion doubling problem", in as each fermionic edge contributes an ordinary integration variables, which essentially account for the denominator q_i in (2.3.1) and (2.3.3), as well as a *pair* of Grassman variables accounting for the numerator q_i in (2.3.1) and (2.3.3).

Let $\mathcal{Q}_f$ denote the $2f \times 2f$ antisymmetric matrix

$$\mathcal{Q}_f = \begin{pmatrix} 0 & \mathrm{q}_1 & 0 & 0 & \cdots & 0 & 0 \\ -\mathrm{q}_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mathrm{q}_2 & \cdots & 0 & 0 \\ 0 & 0 & -\mathrm{q}_2 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \cdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathrm{q}_f \\ 0 & 0 & 0 & 0 & \cdots & -\mathrm{q}_f & 0 \end{pmatrix}. \tag{2.3.4}$$

Lemma 2.6. *Let $\Sigma_{n|2f}$ denote the superspace $\Sigma_n \times \mathbb{A}^{0|2f}$. Then the following identity holds:*

$$\frac{\mathrm{q}_1 \cdots \mathrm{q}_f}{q_1 \cdots q_n} = K_{n,f} \int_{\Sigma_{n|2f}} \frac{dt_1 \cdots dt_{n-1} d\theta_1 \cdots d\theta_{2f}}{(t_1 q_1 + \cdots t_n q_n + \frac{1}{2}\theta^t \mathcal{Q}_f \theta)^{n-f}}, \tag{2.3.5}$$

with

$$K_{n,f} = \frac{2^f (n-1)!}{\prod_{k=1}^{f}(-n+f-k+1)}.$$

Proof. We first show that the following identity holds for integration in the Grassmann variables $\theta = (\theta_1, \ldots, \theta_{2f})$:

$$\int \frac{d\theta_1 \cdots d\theta_{2f}}{(1 + \frac{1}{2}\theta^t \mathcal{Q}_f \theta)^{\alpha}} = \frac{f!}{2^f} \binom{-\alpha}{f} \mathrm{q}_1 \cdots \mathrm{q}_f. \tag{2.3.6}$$

In fact, we expand using the Taylor series

$$(1 + x)^{\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k} x^k$$

and the identity

$$\frac{1}{2}\theta^t \mathcal{Q}_f \theta = \sum_{i=1}^{f} \mathrm{q}_i \theta_{2i-1} \theta_{2i},$$

together with the fact that the degree zero variables $x_i = \theta_{2i-1}\theta_{2i}$ commute, to obtain

$$(1+\frac{1}{2}\theta^t \mathcal{Q}_f \theta)^{-\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (\sum_{i=1}^{f} \mathrm{q}_i \theta_{2i-1}\theta_{2i})^k .$$

The rules of Grassmann integration then imply that only the coefficient of $\theta_1 \cdots \theta_{2f}$ remains as a result of the integration. This gives (2.3.6).

For simplicity of notation, we then write $T = t_1 q_1 + \cdots t_n q_n$, so that we have

$$\int_{\Sigma_{n|2f}} \frac{1}{(t_1 q_1 + \cdots t_n q_n + \frac{1}{2}\theta^t \mathcal{Q}_f \theta)^{n-f}} dt_1 \cdots dt_{n-1}\, d\theta_1 \cdots d\theta_{2f} =$$

$$\frac{f!}{2^f} \binom{-n+f}{f} \mathrm{q}_1 \cdots \mathrm{q}_f \int_{\Sigma_n} T^{-n+f} T^{-f} \; dt_1 \cdots dt_{n-1}$$

$$= \frac{f!}{2^f} \binom{-n+f}{f} \mathrm{q}_1 \cdots \mathrm{q}_f \int_{\Sigma_n} \frac{dt_1 \cdots dt_{n-1}}{(t_1 q_1 + \cdots + t_n q_n)^n} = \frac{f!}{2^f (n-1)!} \binom{-n+f}{f} \frac{\mathrm{q}_1 \cdots \mathrm{q}_f}{q_1 \cdots q_n}.$$

□

2.3.2. Graphs with fermionic legs. Consider now the case of graphs that have both bosonic and fermionic legs. We mimic the procedure described above, but by using both ordinary and Grassmann variables in the process.

We divide the edge indices $i = 1, \ldots, n$ into two sets $i_b = 1, \ldots, n_b$ and $i_f = 1, \ldots, n_f$, with $n = n_b + n_f$, respectively labeling the bosonic and fermionic legs. Consequently, given a choice of a basis for the first homology of the graph, indexed as above by $r = 1, \ldots, \ell$, we replace the matrix η_{ir} of (1.7.5), with a matrix of the form

$$\begin{pmatrix} \eta_{i_f r_f} & \eta_{i_f r_b} \\ \eta_{i_b r_f} & \eta_{i_b r_b} \end{pmatrix}. \tag{2.3.7}$$

Here the loop indices $r = 1, \ldots, \ell$ are at first divided into three sets $\{1, \ldots, \ell_{ff}\}$, labelling the loops consisiting of only fermionic edges, $\{1, \ldots, \ell_{bb}\}$ labelling the loops consisting of only bosonic edges, and the remaning variables $\{1, \ldots, \ell_{bf} = \ell - (\ell_{ff} + \ell_{bb})\}$ for the loops that contain both fermionic and bosonic edges. We then introduce two sets of momentum variables: ordinary variables $\mathrm{s}_{r_b} \in \mathbb{A}^{D|0}$, with $r_b = 1, \ldots, \ell_b = \ell_{bb} + \ell_{bf}$, and Grassmann variables $\sigma_{r_f} \in \mathbb{A}^{0|D}$ with $r_f = 1, \ldots, \ell_f = \ell_{ff} + \ell_{bf}$. That is, we assign to each purely fermionic loop a Grassmann momentum variable, to each purely bosonic loop an ordinary momentum variable, and to the loops containing both fermionic and bosonic legs a pair (s_r, σ_r) of an ordinary and a Grassman variable. In (2.3.7) above we write r_f and r_b, respectively, for the indexing sets of these Grassmann and ordinary variables.

We then consider a change of variables

$$\mathrm{p}_{i_b} = \mathrm{u}_{i_b} + \sum_{r_f} \eta_{i_b r_f} \sigma_{r_f} + \sum_{r_b} \eta_{i_b r_b} \mathrm{s}_{r_b}, \quad \mathrm{p}_{i_f} = \mathrm{u}_{i_f} + \sum_{r_f} \eta_{i_f r_f} \sigma_{r_f} + \sum_{r_b} \eta_{i_f r_b} \mathrm{s}_{r_b}. \tag{2.3.8}$$

analogous to the one used before, where now, for reasons of homogeneity, we need to assume that the η_{ir_f} are of degree one and the η_{ir_b} are of degree zero, since the p_i are even (ordinary) variables.

We apply the change of variables (2.3.8) to the expression

$$\sum_i t_i p_i^2 + \sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\mathrm{p}_{i_f}. \tag{2.3.9}$$

We assume again, as in the purely bosonic case (*cf.* (18.35) of [**90**]), the relations

$$\sum_i t_i \mathrm{u}_i \eta_{ir} = 0$$

for each loop variable $r = r_b$ and $r = r_f$.

We can then rewrite (2.3.9) in the form

$$\sum_i t_i u_i^2 + \sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\mathrm{u}_{i_f}$$

$$+\sum_{r_b, r_b'} (\sum_i t_i \eta_{ir_b}\eta_{ir_b'})\mathrm{s}_{r_b}\mathrm{s}_{r_b'} - \sum_{r_f r_f'} (\sum_i t_i \eta_{ir_f}\eta_{ir_f'})\sigma_{r_f}\sigma_{r_f'}$$

$$+\sum_{r_b r_f} \left((\sum_i t_i \eta_{ir_b}\eta_{ir_f})\mathrm{s}_{r_b}\sigma_{r_f} - \sigma_{r_f}^\tau \mathrm{s}_{r_b}^\tau (\sum_i t_i \eta_{ir_f}\eta_{ir_b}) \right)$$

$$+\sum_{r_b} (\sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\eta_{i_f r_b})\mathrm{s}_{r_b} + \sum_{r_f} (\sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\eta_{i_f r_f})\sigma_{r_f}.$$

Notice the minus sign in front of the quadratic term in the σ_{r_f}, since for order-one variables $\sigma_{r_f}\eta_{ir_f'} = -\eta_{ir_f'}\sigma_{r_f}$. We write the above in the simpler notation

$$T + \mathrm{s}^\tau M_b(t)\mathrm{s} - \sigma^\tau M_f(t)\sigma + \sigma^\tau M_{fb}(t)\mathrm{s} - \mathrm{s}^\tau M_{bf}(t)\sigma + N_b(\theta)\mathrm{s} + N_f(\theta)\sigma, \tag{2.3.10}$$

where τ denotes transposition, $\mathrm{s} = (\mathrm{s}_{r_b})$, $\sigma = (\sigma_{r_f})$, and

$$\begin{aligned} T &= \textstyle\sum_i t_i u_i^2 + \sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\mathrm{u}_{i_f}, \\ M_b(t) &= \textstyle\sum_i t_i \eta_{ir_b}\eta_{ir_b'}, \\ M_f(t) &= \textstyle\sum_i t_i \eta_{ir_f}\eta_{ir_f'} = -M_f(t)^\tau, \\ M_{fb}(t) &= \textstyle\sum_i t_i \eta_{ir_b}\eta_{ir_f}, \\ N_b(\theta) &= \textstyle\sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\eta_{i_f r_b}, \\ N_f(\theta) &= \textstyle\sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\eta_{i_f r_f}. \end{aligned} \tag{2.3.11}$$

Since the η_{i,r_f} are of degree one and the η_{i,r_b} of degree zero, the matrices M_b and M_f are of degree zero, the M_{bf} and M_{fb} of degree one, while the N_b and N_f are, respectively, of degree zero and one. Thus, the expression (2.3.10) is of degree zero. Notice that, since the η_{ir_f} are of order one, the matrix $M_f(t)$ is antisymmetric. We also set $M_{bf}(t) = M_{fb}(t) = M_{fb}(t)^\tau$.

We then consider an integral of the form

$$\int \frac{d^D\mathrm{s}_1 \cdots d^D\mathrm{s}_{\ell_b}\, d^D\sigma_1 \cdots d^D\sigma_{\ell_f}}{(\sum_i t_i p_i^2 + \sum_{i_f} \theta_{2i_f-1}\theta_{2i_f}\mathrm{p}_{i_f})^{n-f}} =$$

$$\int \frac{d^D s_1 \cdots d^D s_{\ell_b}\, d^D\sigma_1 \cdots d^D\sigma_{\ell_f}}{(T + s^\tau M_b(t)s + N_b(\theta)s - \sigma^\tau M_f(t)\sigma + \sigma^\tau M_{fb}(t)s - s^\tau M_{fb}(t)^\tau \sigma + N_f(\theta)\sigma)^{n-f}}, \tag{2.3.12}$$

where the $d^D\sigma_i = d\sigma_{i1}\cdots d\sigma_{iD}$ are Grassmann variables integrations and the $d^D s_i$ are ordinary integrations.

Recall that for Grassmann variables we have the following change of variable formula.

Lemma 2.7. *Suppose given an invertible antisymmetric $N \times N$-matrix A with entries of degree zero and an N-vector J with entries of degree one. Then we have*

$$\sigma^\tau A\sigma + \frac{1}{2}(J^\tau\sigma - \sigma^\tau J) = \eta^\tau A\eta + \frac{1}{4}J^\tau A^{-1}J, \tag{2.3.13}$$

for $\eta = \sigma - \frac{1}{2}A^{-1}J$.

Proof. The result is immediate: since $A^\tau = -A$, we simply have

$$\eta^\tau A\eta = \sigma^\tau A\sigma + \frac{1}{2}J^\tau\sigma - \frac{1}{2}\sigma^\tau J - \frac{1}{4}J^\tau A^{-1}J.$$

□

We then use this change of variable to write

$$\begin{aligned} &-\sigma^\tau M_f(t)\sigma + \sigma^\tau M_{fb}(t)s - s^\tau M_{fb}(t)^\tau\sigma + \tfrac{1}{2}(\sigma^\tau N_f(\theta) - N_f(\theta)^\tau\sigma) = \\ &-\eta^\tau M_f(t)\eta - \tfrac{1}{4}(M_{fb}(t)s + \tfrac{1}{2}N_f(\theta))^\tau M_f(t)^{-1}(M_{fb}(t)s + \tfrac{1}{2}N_f(\theta)) \end{aligned} \tag{2.3.14}$$

with

$$\eta = \sigma - \frac{1}{2}M_f(t)^{-1}\left(M_{fb}(t)s + \frac{1}{2}N_f(\theta)\right). \tag{2.3.15}$$

We have

$$\begin{aligned} &\frac{1}{4}(M_{fb}(t)s + \frac{1}{2}N_f(\theta))^\tau M_f(t)^{-1}(M_{fb}(t)s + \frac{1}{2}N_f(\theta)) = \\ &\frac{1}{4}s^\tau M_{bf}(t)M_f(t)^{-1}M_{fb}(t)s + \frac{1}{8}(N_f(\theta)^\tau M_f(t)^{-1}M_{fb}(t)s + s^\tau M_{bf}(t)M_f(t)^{-1}N_f(\theta)) \\ &+\frac{1}{16}N_f(\theta)^\tau M_f(t)^{-1}N_f(\theta). \end{aligned}$$

We then let

$$U(t,\theta,s) := T + C(t,\theta) + s^\tau A_b(t)s + B_b(t,\theta)s, \tag{2.3.16}$$

where

$$\begin{aligned} A_b(t) &= M_b(t) - \tfrac{1}{4}M_{bf}(t)M_f(t)^{-1}M_{fb}(t) \\ B_b(t,\theta) &= N_b(\theta) - \tfrac{1}{4}N_f(\theta)^\tau M_f(t)^{-1}M_{fb}(t) \\ C(t,\theta) &= -\tfrac{1}{16}N_f(\theta)^\tau M_f(t)^{-1}N_f(\theta). \end{aligned} \tag{2.3.17}$$

Thus, we write the denominator of (2.3.12) in the form

$$U(t,\theta,s)^{n-f}\left(1 + \frac{1}{2}\eta^\tau X_f(t,\theta,s)\eta\right)^{n-f}, \tag{2.3.18}$$

where we use the notation

$$X_f(t,\theta,\mathrm{s}) := 2U(t,\theta,\mathrm{s})^{-1}M_f(t). \tag{2.3.19}$$

Thus, the Grassmann integration in (2.3.12) gives, as in Lemma 2.6,

$$\int \frac{d^D\eta_1\cdots d^D\eta_{\ell_f}}{\left(1+\frac{1}{2}\eta^\tau X_f(t,\theta,\mathrm{s})\eta\right)^{n-f}} = C_{n,f,\ell_f}\,\frac{2^{D\ell_f/2}}{U(t,\theta,\mathrm{s})^{D\ell_f/2}}\det(M_f(t))^{D/2}, \tag{2.3.20}$$

where C_{n,f,ℓ_f} is a combinatorial factor obtained as in Lemma 2.6.

We then proceed to the remaining ordinary integration in (2.3.12). We have, dropping a multiplicative constant,

$$\det(M_f(t))^{D/2}\int \frac{d^D\mathrm{s}_1\cdots d^D\mathrm{s}_{\ell_b}}{U(t,\theta,\mathrm{s})^{n-f+D\ell_f/2}}. \tag{2.3.21}$$

This now can be computed as in the original case we reviewed in §2.3 above. We use the change of variables $\mathrm{v}=\mathrm{s}+\frac{1}{2}M_b(t)^{-1}N_b(\theta)^\tau$. We then have

(2.3.22)

$$\mathrm{v}^\tau A_b(t)\mathrm{v} = \mathrm{s}^\tau A_b(t)\mathrm{s} + \frac{1}{2}\mathrm{s}^\tau B_b(t,\theta)^\tau + \frac{1}{2}B_b(t,\theta)\mathrm{s} + \frac{1}{4}B_b(t,\theta)A_b(t)^{-1}B_b(t,\theta)^\tau,$$

where $A_b(t)^\tau = A_b(t)$ and $(B_b(t,\theta)\mathrm{s})^\tau = B_b(t,\theta)\mathrm{s}$.

We then rewrite (2.3.21) in the form

$$\det(M_f(t))^{D/2}\int \frac{d^D\mathrm{v}_1\cdots d^D\mathrm{v}_{\ell_b}}{(T+C-\frac{1}{4}B_bA_b^{-1}B_b^\tau+\mathrm{v}^\tau A_b\mathrm{v})^{n-f+D\ell_f/2}}. \tag{2.3.23}$$

Set then

$$\tilde{T}(t,\theta) = T(t,\theta)+C(t,\theta)-\frac{1}{4}B_b(t,\theta)A_b^{-1}(t)B_b(t,\theta)^\tau, \tag{2.3.24}$$

so that we write the above as

$$\frac{\det(M_f(t))^{D/2}}{\tilde{T}(t,\theta)^{n-f+D\ell_f/2}}\int \frac{d^D\mathrm{v}_1\cdots d^D\mathrm{v}_{\ell_b}}{(1+\mathrm{v}^\tau X_b(t,\theta)\mathrm{v})^{n-f+D\ell_f/2}},$$

with

$$X_b(t,\theta) = \tilde{T}(t,\theta)^{-1}A_b(t).$$

Then, up to a multiplicative constant, the integral gives

$$\tilde{T}^{-n+f-\frac{D\ell_f}{2}+\frac{D\ell_b}{2}}\,\frac{\det(M_f(t))^{D/2}}{\det(A_b(t))^{D/2}}. \tag{2.3.25}$$

Consider first the term

$$\frac{\det(M_f(t))^{D/2}}{\det(A_b(t))^{D/2}}$$

in (2.3.25) above. This can be identified with a Berezinian. In fact, we have

$$\frac{\det(M_f(t))^{D/2}}{\det(M_b(t)-\frac{1}{4}M_{fb}(t)M_f(t)^{-1}M_{fb}(t))^{D/2}} = \mathrm{Ber}(\mathcal{M}(t))^{-D/2}, \tag{2.3.26}$$

where

$$\mathcal{M}(t) = \begin{pmatrix} M_b(t) & \frac{1}{2}M_{fb}(t) \\ \frac{1}{2}M_{bf}(t) & M_f(t)\end{pmatrix}. \tag{2.3.27}$$

We now look more closely at the remaining term $\tilde{T}^{-n+f-\frac{D\ell_f}{2}+\frac{D\ell_b}{2}}$ in (2.3.25). We know from (2.3.24), (2.3.17), and (2.3.11) that we can write $\tilde{T}(t,\theta)$ in the form

$$\tilde{T}(t,\theta) = \sum_i u_i^2 t_i + \sum_j \mathrm{u}_i \theta_{2j-1}\theta_{2j} + \sum_{i<j} C_{ij}(t)\theta_{2i-1}\theta_{2i}\theta_{2j-1}\theta_{2j}, \tag{2.3.28}$$

where the first sum is over all edges and the other two sums are over fermionic edges. We set $\lambda_i = \theta_{2i-1}\theta_{2i}$. Using a change of variables $\tilde{\lambda}_i = \lambda_i + \frac{1}{2}C\mathrm{u}$, we rewrite the above as

$$\tilde{T}(t,\theta) = \sum_i u_i^2 t_i - \frac{1}{4}\mathrm{u}^\tau C\mathrm{u} + \sum_{i<j} C_{ij}\eta_{2i-1}\eta_{2i}\eta_{2j-1}\eta_{2j},$$

with $\tilde{\lambda}_i = \eta_{2i-1}\eta_{2i}$. We denote by

$$\hat{T}(t) = \sum_i u_i^2 t_i - \frac{1}{4}\mathrm{u}^\tau C\mathrm{u}$$

and we write

$$\tilde{T}^{-\alpha} = \hat{T}^{-\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{\frac{1}{2}\tilde{\lambda}^\tau C\tilde{\lambda}}{\hat{T}}\right)^k$$

where we use the notation $\frac{1}{2}\tilde{\lambda}^\tau C\tilde{\lambda} = \sum_{i<j} C_{ij}\eta_{2i-1}\eta_{2i}\eta_{2j-1}\eta_{2j}$.

Thus, we can write the Feynman integral in the form

$$\int \frac{\mathrm{q}_1\cdots\mathrm{q}_f}{q_1\cdots q_n} d^D s_1\cdots d^D s_{\ell_b}\, d^D\sigma_1\cdots d^D\sigma_{\ell_f} =$$

$$\kappa\int_{\Sigma_{n|2f}} \frac{\Lambda(t)\eta_1\cdots\eta_{2f}}{\hat{T}(t)^{n-\frac{f}{2}+\frac{D}{2}(\ell_f-\ell_b)}\mathrm{Ber}(\mathcal{M}(t))^{D/2}} dt_1\cdots dt_n\, d\eta_1\cdots d\eta_{2f}, \tag{2.3.29}$$

where $\Lambda(t)$ is $\hat{T}^{f/2}$ times the coefficient of $\eta_1\cdots\eta_{2f}$ in the expansion

$$\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(\frac{\frac{1}{2}\tilde{\lambda}^\tau C\tilde{\lambda}}{\hat{T}}\right)^k.$$

More explicitly, this term is of the form

$$\Lambda(t) = \sum C_{i_1 i_2}(t)\cdots C_{i_{f-1}i_f}(t),$$

over indices i_a with $i_{2a-1} < i_{2a}$ and for $k = f/2$. The multiplicative constant in front of the integral on the right hand side above is given by

$$\kappa = \binom{-n+f-\frac{D}{2}(\ell_f-\ell_b)}{f/2}.$$

We then obtain the following result.

THEOREM 2.8. *Suppose given a graph Γ with n edges, of which f fermionic and $b = n - f$ bosonic. Assume that there exists a choice of a basis for $H_1(\Gamma)$ satisfying the condition*

$$n - \frac{f}{2} + \frac{D}{2}(\ell_f - \ell_b) = 0. \tag{2.3.30}$$

Then the following identity holds:

$$\int \frac{\mathrm{q}_1 \cdots \mathrm{q}_f}{q_1 \cdots q_n} d^D s_1 \cdots d^D s_{\ell_b} \, d^D \sigma_1 \cdots d^D \sigma_{\ell_f} = \int_{\Sigma_n} \frac{\Lambda(t)}{\mathrm{Ber}(\mathcal{M}(t))^{D/2}} dt_1 \cdots dt_n. \tag{2.3.31}$$

Proof. This follows directly from (2.3.29), after imposing $n - \frac{f}{2} + \frac{D}{2}(\ell_f - \ell_b) = 0$ and performing the Grassmann integration of the resulting term

$$\int_{\Sigma_{n|2f}} \frac{\Lambda(t)\eta_1 \cdots \eta_{2f}}{\mathrm{Ber}(\mathcal{M}(t))^{D/2}} dt_1 \cdots dt_n \, d\eta_1 \cdots d\eta_{2f}. \tag{2.3.32}$$

□

2.3.3. Graph supermanifolds. The result of the previous section shows that we have an analog of the period integral

$$\int_{\Sigma_n} \frac{dt_1 \cdots dt_n}{\det(M_\Gamma(t))^{D/2}}$$

given by the similar expression

$$\int_{\Sigma_n} \frac{\Lambda(t)}{\mathrm{Ber}(\mathcal{M}(t))^{D/2}} dt_1 \cdots dt_n. \tag{2.3.33}$$

Again we see that, in this case, divergences arise from the intersections between the domain of integration given by the simplex Σ_n and the subvariety of $\mathbb{P}^{n-1}$ defined by the solutions of the equation

$$\frac{\mathrm{Ber}(\mathcal{M}(t))^{D/2}}{\Lambda(t)} = 0. \tag{2.3.34}$$

Lemma 2.9. *For generic graphs, the set of zeros of* (2.3.34) *defines a hypersurface in* $\mathbb{P}^n$*, hence a divisor in* $\mathbb{P}^{n-1|2f}$ *of dimension* $(n-2|2f)$*. The support of this divisor is the same as that of the principal divisor defined by* $\mathrm{Ber}(\mathcal{M}(t))$*.*

Proof. The generic condition on graphs is imposed to avoid the cases with $M_f(t) \equiv 0$. Thus, suppose given a pair (Γ, B) that is generic, in the sense that $M_f(t)$ is not identically zero. The equation (2.3.34) is satisfied by solutions of

$$\det(M_b(t) - \frac{1}{4} M_{bf}(t) M_f(t)^{-1} M_{fb}(t)) = 0$$

and by poles of $\Lambda(t)$. Using the formulae (2.3.17) and (2.3.11) we see that the denominator of $\Lambda(t)$ is given by powers of $\det(M_f(t))$ and $\det(A_b(t)) = \det(M_b(t) - \frac{1}{4} M_{bf}(t) M_f(t)^{-1} M_{fb}(t))$. Thus, the set of solutions of (2.3.34) is the union of zeros and poles of $\mathrm{Ber}(\mathcal{M}(t))$. The multiplicities are given by the powers of these determinants that appear in $\Lambda(t)\mathrm{Ber}(\mathcal{M}(t))^{-D/2}$. □

Definition 2.10. Let Γ be a graph with bosonic and fermionic edges and B a choice of a basis of $H_1(\Gamma)$. We denote by $\mathcal{X}_{(\Gamma,B)} \subset \mathbb{P}^{n-1|2f}$ the locus of zeros and poles of $\mathrm{Ber}(\mathcal{M}(t)) = 0$. We refer to $\mathcal{X}_{(\Gamma,B)}$ as the *graph supermanifold*.

In the degenerate cases of graphs such that $M_f(t) \equiv 0$, we simply set $\mathcal{X}_{(\Gamma,B)} = \mathbb{P}^{n-1|2f}$. Examples of this sort are provided by data (Γ, B) such that there is only one loop in B containing fermionic edges. Other special cases arise when we

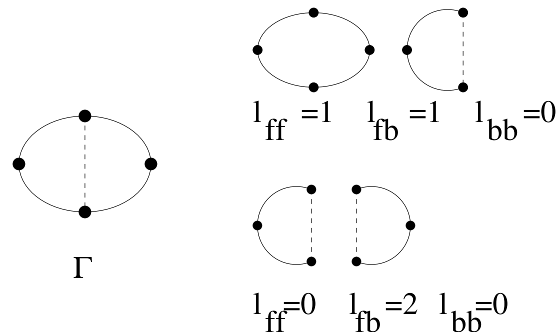


FIGURE 1. Choices of a basis for $H_1(\Gamma)$.

consider graphs with only bosonic or only fermionic edges. In the first case, we go back to the original calculation without Grassmann variables and we therefore simply recover $\mathcal{X}_{(\Gamma,B)} = X_\Gamma = \{t : \det(M_b(t)) = 0\} \subset \mathbb{P}^{n-1|0}$. In the case with only fermionic edges, we have $\det(M_b(t) - \frac{1}{4}M_{bf}(t)M_f(t)^{-1}M_{fb}(t)) \equiv 0$ since both $M_b(t)$ and $M_{bf}(t)$ are identically zero. It is then natural to simply assume that, in such cases, the graph supermanifold is simply given by $\mathcal{X}_{(\Gamma,B)} = \mathbb{P}^{f-1|2f}$.

2.3.4. Examples from Feynman graphs. We still need to check that the condition (2.3.30) we imposed on the graph is satisfied by some classes of interesting graphs. First of all, notice that the condition does not depend on the graph alone, but on the choice of a basis for $H_1(\Gamma)$. The same graph can admit choices for which (2.3.30) is satisfied and others for which it fails to hold. For example, consider the graph illustrated in Figure 1, for a theory in dimension $D = 6$, where we denoted bosonic edges by the dotted line and fermionic ones by the full line. There exists a choice of a basis of $H_1(\Gamma)$ for which (2.3.30) is satisfied, as the first choice in the figure shows, while not all choices satisfy this condition, as one can see in the second case.

One can see easily that one can construct many examples of graphs that admit a basis of $H_1(\Gamma)$ satisfying (2.3.30). For instance, the graph in Figure 2 is a slightly more complicated example in $D = 6$ of a graph satisfying the condition. Again we used dotted lines for the bosonic edges and full lines for the fermionic ones.

Let us consider again the example of the very simple graph of Figure 1, with the first choice of the basis B for $H_1(\Gamma)$. This has two generators, one of them a loop made of fermionic edges and the second a loop containing both fermionic and bosonic edges. Let us assign the ordinary variables t_i with $i = 1, \ldots, 5$ to the edges as in Figure 3. We then have

$$M_b(t) = t_1 + t_2 + t_3$$

since only the second loop in the basis contains bosonic edges, while we have

$$M_{bf}(t) = (t_1 + t_2, t_1 + t_2 + t_3) = t_1(1,1) + t_2(1,1) + t_3(0,1) + t_4(0,0) + t_5(0,0)$$

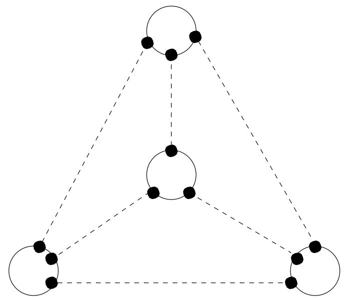


FIGURE 2. A graph with a basis of $H_1(\Gamma)$ satisfying (2.3.30).

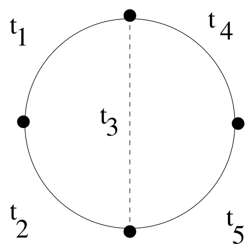


FIGURE 3. Edge variables.

and

$$M_f(t) = \begin{pmatrix} 0 & t_1+t_2 \\ -(t_1+t_2) & 0 \end{pmatrix}.$$

Thus, we obtain in this case

$$M_{bf}(t)M_f(t)^{-1}M_{fb}(t) = (t_1+t_2, t_1+t_2+t_3) \begin{pmatrix} 0 & \frac{-1}{t_1+t_2} \\ \frac{1}{t_1+t_2} & 0 \end{pmatrix} \begin{pmatrix} t_1+t_2 \\ t_1+t_2+t_3 \end{pmatrix}$$

$$= (t_1+t_2, t_1+t_2+t_3) \begin{pmatrix} \frac{-(t_1+t_2+t_3)}{t_1+t_2} \\ 1 \end{pmatrix} = -(t_1+t_2+t_3) + t_1+t_2+t_3 \equiv 0.$$

Thus, in this particular example we have $M_{bf}(t)M_f(t)^{-1}M_{fb}(t) \equiv 0$ for all $t = (t_1, \dots, t_5)$, so that $\mathrm{Ber}(\mathcal{M}(t)) = \det(M_b(t))\det(M_f(t))^{-1} = (t_1+t_2+t_3)/(t_1+t_2)^2$ and the locus of zeros and poles $\mathcal{X}_{(\Gamma,B)} \subset \mathbb{P}^{5|8}$ is the union of $t_1+t_2+t_3=0$ and $t_1+t_2=0$ in $\mathbb{P}^5$ (the latter counted with multiplicity two), with the restriction of the sheaf from $\mathbb{P}^{5|8}$.

2.3.5. The universality property. Lemma 2.11 below shows to what extent the "universality property" of graph hypersurfaces, *i.e.* the fact that they generate the Grothendieck group of varieties, continues to hold when passing to supermanifolds.

LEMMA 2.11. *Let $\mathcal{R}$ be the subring of the Grothendieck ring $K_0(\mathcal{SV}_{\mathbb{C}})$ of supermanifolds spanned by the $[\mathcal{X}_{(\Gamma,B)}]$, for $\mathcal{X}_{(\Gamma,B)}$ the graph supermanifolds defined*

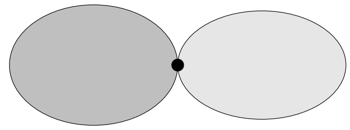


FIGURE 4. Graphs with $\ell_{bf} = 0$.

by the divisor of zeros and poles of the Berezinian $\mathrm{Ber}(\mathcal{M}(t))$*, with* B *a choice of a basis of* $H_1(\Gamma)$*. Then*

$$\mathcal{R} = K_0(\mathcal{V}_{\mathbb{C}})[T^2] \subset K_0(\mathcal{S}\mathcal{V}_{\mathbb{C}}),$$

where $T = [\mathbb{A}^{0|1}]$.

PROOF. By Corollary 2.4 and the universality result of [**87**], it suffices to prove that the subring of $K_0(\mathcal{S}\mathcal{V}_{\mathbb{C}})$ generated by the $[\mathcal{X}_{(\Gamma,B)}]$ contains the classes of the ordinary graph hypersurfaces in $K_0(\mathcal{V}_{\mathbb{C}})$ and the class $[\mathbb{A}^{0|2}]$.

To show that $\mathcal{R}$ contains the ordinary graph hypersurfaces, consider the special class of graphs that are of the form schematically illustrated in Figure 4. These are unions of two graphs, one only with bosonic edges and one only with fermionic edges, with a single vertex in common. Notice that in actual physical theories the combinatorics of graphs with only fermionic edges is severely restricted (see [**104**], §5.3) depending on the dimension D in which the theory is considered. However, for the purpose of this universality result, we allow arbitrary D and corresponding graphs, just as in the result of [**87**] one does not restrict to the Feynman graphs of any particular theory.

The graphs of Figure 4 provide examples of graphs with bases of $H_1(\Gamma)$ containing loops with only fermionic or only bosonic legs, *i.e.* with $\ell_{bf} = 0$, $\ell_f = \ell_{ff}$ and $\ell_b = \ell_{bb}$. This implies that, for all these graphs $\Gamma = \Gamma_B \cup_v \Gamma_F$ with the corresponding bases of H_1, one has $M_{bf}(t) \equiv 0$, since for each edge variable t_i one of the two factors $\eta_{ir_b}\eta_{ir_f}$ is zero. Thus, for this class of examples we have $\mathrm{Ber}(\mathcal{M}(t)) = \det(M_b(t))/\det(M_f(t))$. Moreover, we see that for these examples $\det(M_b(t)) = \Psi_{\Gamma_b}(t)$ is the usual graph polynomial of the graph Γ_B with only bosonic edges. Since such Γ_B can be any arbitrary ordinary graph, we see that the locus of zeros alone, and just for this special subset of the possible graphs, already suffices to generate the full $K_0(\mathcal{V}_{\mathbb{C}})$ since it gives all the graph varieties $[X_{\Gamma_B}]$.

To show then that the subring $\mathcal{R}$ contains the classes $[\mathbb{A}^{0|2f}]$, for all f, first notice that the classes $[\mathbb{P}^n][\mathbb{A}^{0|2f}] = [pt][\mathbb{A}^{0|2f}] + [\mathbb{A}^{1|0}][\mathbb{A}^{0|2f}] + \cdots + [\mathbb{A}^{n|0}][\mathbb{A}^{0|2f}]$ belong to $\mathcal{R}$, for all n and f. These are supplied, for instance, by the graphs with a single loop containing fermionic edges, as observed above. This implies that elements of the form $[\mathbb{A}^{n|0}][\mathbb{A}^{0|2f}] = [\mathbb{P}^n][\mathbb{A}^{0|2f}] - [\mathbb{P}^{n-1}][\mathbb{A}^{0|2f}]$ belong to $\mathcal{R}$. In particular the graph consisting of a single fermionic edge closed in a loop gives $[\mathbb{A}^{0|2f}]$ in $\mathcal{R}$. ∎

Notice that in [**87**], in order to prove that the correspoding graph hypersurfaces generate $K_0(\mathcal{V}_{\mathbb{C}})$, one considers all graphs and not only the log divergent ones with

$n = D\ell/2$, even though only for the log divergent ones the period has the physical interpretation as Feynman integral. Similarly, here, in Lemma 2.11, we consider all (Γ, B) and not just those satisfying the condition (2.3.30).

The fact that we only find classes of the even dimensional superplanes $[\mathbb{A}^{0|2f}]$ in $\mathcal{R}$ instead of all the possible classes $[\mathbb{A}^{0|f}]$ is a consequence of the *fermion doubling* used in Lemma 2.6 in the representation of the Feynman integral in terms of an ordinary and a fermionic integration.

2.4. Supermanifolds and mirrors

We discuss here some points of contact between the construction we outlined in this paper and the supermanifolds and periods that appear in the theory of mirror symmetry.

Supermanifolds arise in the theory of mirror symmetry (see for instance [**107**], [**86**], [**96**]) in order to describe mirrors of rigid Calabi–Yau manifolds, where the lack of moduli of complex structures prevents the existence of Kähler moduli on the mirror. The mirror still exists, not as a conventional Kähler manifold, but as a supermanifold embedded in a (weighted) super-projective space.

For instance, in the construction given in [**107**], one considers the hypersurface in (weighted) projective space given by the vanishing of a superpotential $X = \{W = 0\} \subset \mathbb{P}^n$. The local ring of the hypersurface X is given by polynomials in the coordinates modulo the Jacobian ideal $\mathcal{R}_X = \mathbb{C}[x_i]/dW(x_i)$. To ensure the vanishing of the first Chern class, one corrects the superpotential W by additional quadratic terms in either bosonic or fermionic variables, so that the condition $W = 0$ defines a supermanifold embedded in a (weighted) super-projective space, instead of an ordinary hypersurface in projective space.

In the ordinary case, one obtains the primitive part of the middle cohomology $H_0^{n-1}(X)$ and its Hodge decomposition via the Poincaré residue

$$Res(\omega) = \int_C \omega, \tag{2.4.1}$$

with C a 1-cycle encircling the hypersurface X, applied to forms of the form

$$\omega(P) = \frac{P(x_0, \dots, x_n)\Omega}{W^k}, \tag{2.4.2}$$

with $\Omega = \sum_{i=0}^n (-1)^i \lambda_i x_i dx_0 \cdots \widehat{dx_i} \cdots dx_n$, as in (1.7.17) with λ_i the weights in the case of weighted projective spaces, and with $P \in \mathcal{R}_X$ satisfying $k \deg(W) = \deg(P) + \sum_i \lambda_i$.

In the supermanifold case, one replaces the calulation of the Hodge structure on the mirror done using the technique described above, by a supergeometry analog, where the forms (2.4.2) are replaced by forms

$$\frac{P(x_0, \dots, x_n) d\theta_1 \cdots d\theta_{2m}\Omega}{W^k}, \tag{2.4.3}$$

where here the superpotential W is modified by the presence of an additional quadratic term in the fermionic variables $\theta_1\theta_2 + \cdots \theta_{2m-1}\theta_{2m}$.

In comparison to the setting discussed in this paper, notice that the procedure of replacing the potential W by $W' = W + \theta_1\theta_2 + \cdots \theta_{2m-1}\theta_{2m}$, with the additional fermionic integration, is very similar to the first step in our derivation where we replaced the original expression $T = t_1 q_1 + \cdots t_n q_n$ by the modified one $T + \frac{1}{2}\theta^\tau \mathcal{Q}\theta$ with $\frac{1}{2}\theta^\tau \mathcal{Q}\theta = \mathrm{q}_1\theta_1\theta_2 + \cdots + \mathrm{q}_f\theta_{2f-1}\theta_{2f}$. Thus, replacing the ordinary integration $\int T^{-n}(t)dt$ by the integration $\int (T(t) + \frac{1}{2}\theta^\tau \mathcal{Q}\theta)^{-n+f} dtd\theta$ is an analog of replacing the integral $\int W^{-k}dt$ with the integral $\int (W + \theta_1\theta_2 + \cdots \theta_{2m-1}\theta_{2m})^{-k} dtd\theta$ used in the mirror symmetry context. However, there seems to be no analog, in that setting, for the type of periods of the form (2.3.33) that we obtain here and for the corresponding type of supermanifolds defined by divisors of Berezinians considered here.

CHAPTER 3

Graph insertions and hypersurface singularities

This Chapter is the result of my joint work with Christoph Bergbauer.

3.1. Connes-Kreimer theory

Recall the existence of a Hopf algebra structure on the space of Feynman graphs of a given physical theory of Feynman graphs associated to a given quantum field theory (cf. [**15**].) The coproduct on this Hopf algebra– the *Connes–Kreimer Hopf algebra–* is given by

$$\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \tag{3.1.1}$$

where $\gamma \subset \Gamma$ are (one-particle irreducible) subgraphs of a given Feynman graph Γ and the quotient Γ/γ denotes a graph obtained by contracting each component of γ into a vertex. The reader is referred to definitions 1.12 and 1.14 of [**18**] for a very careful description of the subgraphs allowed in (3.1.1).

The BPHZ recursion formula for renormalization can be conceptually understood through this Hopf algebra as a problem of Birkhoff factorization of loops on the associated affine group scheme and as a Riemann-Hilbert problem. In [**17**] it is shown that using viewpoint one can construct a Tannakian category of a geometric origin with the associated motivic Galois group containing the renormalization group as a 1-parameter subgroup. (See chapter 1 of [**18**] for a recent and extensive review of these and related topics.)

Through a general argument (the Milnor-Moore theorem), one can also construct the Connes-Kreimer Hopf algebra from a certain Lie algebra whose bracket is given by the insertion of one graph Γ into another graph Γ' at the vertices. More explicitly, the Lie bracket is given by the formula (see [**16**])

$$[\Gamma, \Gamma'] = \Big(\sum_{\text{all vertices of } \Gamma} \Gamma \leftarrow \Gamma' \Big) - \Big(\sum_{\text{all vertices of } \Gamma'} \Gamma' \leftarrow \Gamma \Big), \tag{3.1.2}$$

where $\Gamma \leftarrow \Gamma'$ denotes the insertion of Γ' at a given vertex of Γ and similiarly for $\Gamma' \leftarrow \Gamma$.

Feynman graphs form a category **FGph** of Feynman graph [**116**] where the objects of **FGph** are labeled Feynman graphs (with a distinguished empty graph) and the set of morphisms given by

$$\mathrm{Hom}(\Gamma_1, \Gamma_2) := \{(\gamma_1, \gamma_2, f) : f : \Gamma_1/\gamma_1 \simeq \gamma_2\}.$$

Here the quotient Γ_1/γ_1 is the standard quotient for Γ_1 connected and the isomorphism means a bijection between the sets of vertices and edges of Γ_1 and Γ_2 inducing bijections on incidences. It is shown in [**116**] that **FGph** is a symmetric monodial catgory (but not abelian) with $\mathrm{Hom}(\Gamma_1, \Gamma_2)$ and $\mathrm{Ext}^n(\Gamma_1, \Gamma_2)$ being finite sets. The category **FGph** resembles a finitary abelian category and the main result of [**116**] is that the notion of a Ringel-Hall algebra makes sense for **FGph** and furthermore, the Connes-Kreimer Lie algebra ([**16**]) can be viewed as one. It is worth remarking that the Grothendieck group of this category is

$$K(\mathbf{FGph}) = \mathbb{Z}[\mathcal{P}] \tag{3.1.3}$$

of primitive graphs.

3.2. Insertion and graph polynomials

Most of what would follow is a study of insertions into vertices (3.1.2) even though for the sake of completeness we also formulate the (uninteresting) case of insertion into edges.

3.2.1. Notation, preliminaries and comments. All through we work with a fixed gluing map (one-to-one) $s : E_v \to E_\gamma^{ext}$ once and for all. For a given graph Γ, we write E_Γ to mean the set of edges of and E_Γ^{ext} to mean the set of half/external edges of Γ. The set of vertices of Γ is denoted as V_Γ. The complete graph with n vertices is denoted by F_n. The notion of partition of a set is the usual one and we use the terminology *subordinate* partition to mean the usual *refinement* of a partition; all other terminology and properties of partitions are applicable in our case.

Let $\gamma \subset \Gamma$ be a tree i.e. $h_1(\gamma) = 0$ in a connected graph Γ with n internal edges. Let $\gamma \subset \Gamma$ be a tree i.e. $h_1(\gamma) = 0$ in a connected graph Γ with n internal edges. In terms of the embedding $\mathbb{P}(E_\Gamma) \hookrightarrow \mathbb{P}^{\#E_\Gamma - 1}$ we have the following two results about the relationship between X_Γ, X_γ and $X_{\Gamma/\gamma}$.

Proposition 3.1 (proposition 3.4 of [**92**]). *Let $X_\Gamma \subset \mathbb{P}(E_\Gamma)$ and let $L(\gamma) = \bigcup_{e \in E_\gamma} \{A_e = 0\}$ be the linear subspace of $\mathbb{P}(E_\Gamma)$ corresponding to γ. Then $L(\gamma)$ is identified with $\mathbb{P}(E_{\Gamma/\gamma})$ and under this identification*

$$X_{\Gamma/\gamma} = X_\Gamma \cap L(\gamma).$$

Now let us assume that $h_1(\gamma) > 0$. Then we know that $L(\gamma)$ is contained in X_Γ (proposition 3.1 of [**92**].) In this case we have the following proposition about the blowups of coordinate linear subspaces associated to the subgraphs γ and their identification with X_γ and $X_{\Gamma/\gamma}$.

Proposition 3.2 (proposition 3.5 of [**92**]). *Let $P \to \mathbb{P}(E_\Gamma)$ be the blowup of $L(\gamma) \subset \mathbb{P}(E_\Gamma)$ and let $F \subset P$ be its exceptional locus. Let $Y \subset P$ be the strict transform of X_Γ in P. Then there are the following canonical identifications*

$$\begin{aligned} F &\simeq \mathbb{P}(E_\Gamma) \times \mathbb{P}(E_{\Gamma/\gamma}), \\ Y \cap F &= (X_\gamma \times \mathbb{P}(E_{\Gamma/\gamma})) \cup (\mathbb{P}(E_\gamma) \times X_{\Gamma/\gamma}). \end{aligned}$$

The proof of proposition 3.2 relies on the identification of the polynomial ring $K[A_e]_{e \in \Gamma/\gamma} \otimes K[A_e]_{e \in \gamma}$ with the tensor product of the coordinate rings of $\mathbb{P}(E_{\Gamma/\gamma})$ and $\mathbb{P}(E_\gamma)$ and then showing that $\Psi_{\Gamma/\gamma}(A_e)_{e \in \Gamma/\gamma} \cdot \Psi_\gamma(A_e)_{e \in \gamma} \in K[A_e]_{e \in \Gamma/\gamma} \otimes K[A_e]_{e \in \gamma}$.

As clarified in the remark after proposition 6.3 in [**8**], proposition 3.2 is the geometric interpretation of the formula $\Psi_\Gamma = \Psi_\gamma \Psi_{\Gamma/\gamma} + f(A_1, \ldots, A_m)$ where m = number of internal edges of the quotient graph Γ/γ and $f(A_1, \ldots, A_m)$ is some polynomial of degree strictly greater than $h_1(\gamma)$. Our main result would be an explicit identification of the term $f(A_1, \ldots, A_m)$ (which we call the *direct* graph polynomial) in the context of graph insertions instead of the dual operation of taking quotients by subgraphs as in the coproduct (3.1.1).

3.2.2. Insertion into vertices. Let γ and Γ be two graphs and $v \in V_\Gamma$. Let E_v be the set of edges in $E_\Gamma \cup E_\Gamma^{ext}$ adjacent to v. Let $\Gamma \leftarrow \gamma$ be the graph obtained by removing v and identifying external (open) edges via s. Our main problem is to write $\Psi_{\Gamma \leftarrow \gamma}$ in terms of graph polynomials of Γ and γ.

Let $\overline{\Gamma - v}$ be the graph Γ where $v \in E_\Gamma$ has been split into $\#E_v$ disjoint pieces. These $\#E_v$ vertices are called *split vertices*.

Consider now on E_v the following equivalence relation: $e_1 \sim e_2$ iff e_1 and e_2 are connected in $\overline{\Gamma - v}$. and write P_v to denote the resulting partition of E_v. Note the following:

REMARK 3.3. For $e_1 \sim e_2$ it is necessary (but not sufficient) that there is a cycle $c \in H_1(\Gamma)$ with $e_1 \cup e_2 \subset c$.

Let P be subordinate to P_v. For an external line $e \in E_\gamma^{ext}$ denote by ∂e the unique (external) vertex of γ which meets e. Now for all $\{q_1 \ldots q_n\} \in P$ identify (merge) the vertices $\partial s(q_1) \ldots \partial s(q_n)$ of γ respectively. The resulting graph is denoted γ_P. Note that $E_{\gamma_P} = E_\gamma$ in our conventions.

Recall the complete graph F_n where $n = \#E_v$. Fix a bijection $b : E_v \to E_{F_n}^{ext}$. The partition P of E_v then induces a partition of V_{F_n} which we denote by $P' = \partial b P$. Let now d be a spanning tree of F_n such that all restrictions of d to the full subgraphs of F_n defined by any $Q \in P'$ are connected. Consider the graph $\Gamma \leftarrow_{v,b} F_n$ and remove from it all edges of d which connect vertices from the same cell of P'. Then shrink (collapse) all edges of d which connect different cells of P'. The resulting graph will depend in general on the choice of d and is denoted $\Gamma^P(d)$. For the proof of an intermediate proposition, we would need the following lemma:

LEMMA 3.4. *Let G be a graph and $G_1, \ldots, G_n$ a chain of subgraphs, i. e. $G_i \cap G_{i+1} = \{v_i\}$ with $v_i \in V_G$, $v_i \neq v_j$ for $i \neq j$, and $G_i \cap G_j = \emptyset$ for $j \neq i, i+1$. Then*

$$\Psi_{G_1 \cup \ldots \cup G_n} = \prod_i \Psi_{G_i}.$$

Another useful notion would be the *direct* spanning tree:

DEFINITION 3.5. Let $P \leq P_v$ and $\Gamma^P(d)$ as defined above. Let

$$\tilde{\Psi}_{\Gamma^P}(d) := \sum_t \prod_{e \notin t} A_e$$

where the sum is over all spanning trees t of $\Gamma^P(d)$ such that for each $Q \in P$ and any edges $e_1, e_2 \in Q$ the (unique) path in t from e_1 to e_2 does not meet any of the edges in $E_v \setminus Q$. We call such a t a *direct* spanning tree and $\tilde{\Psi}_{\Gamma^P}(d)$ the *direct* graph polynomial.

PROPOSITION 3.6. *The direct graph polynomial is independent of d and thus simply denoted as $\tilde{\Psi}_{\Gamma^P}$.*

PROOF. Let t be a direct spanning tree. For $Q \in P$ let t_Q be the subgraph of t defined as the union of paths from q_1 to q_2 where q_1, q_2 vary in Q. Then t_Q is again a spanning tree as a consequence of t being direct. Let $P = \{Q_1, \dots, Q_n\}$. By the lemma 3.4, $\Psi_{t_{Q_1} \cup \dots \cup t_{Q_n}} = \prod_i \Psi_{t_{Q_i}} = \prod_i \prod_{e \notin t_{Q_i}} A_e$ which is independent of d. This implies that the sum $\tilde{\Psi}_{\Gamma^P} = \sum_t \Psi_{t_{Q_1} \cup \dots \cup t_{Q_n}}$. □

We now state and prove the main theorem of this Chapter:

THEOREM 3.7. *The graph polynomial of $\Gamma \leftarrow \gamma$ is determined as follows.*

(1)

$$\Psi_{\Gamma \leftarrow \gamma} = \Psi_\gamma \Psi_\Gamma + \tilde{\Psi}_{\gamma,\Gamma}$$

where $\tilde{\Psi}_{\gamma,\Gamma}$, unless vanishing, is of degree higher (resp. lower) in the edge variables of γ (resp. Γ) and

(2) *$\tilde{\Psi}_{\gamma,\Gamma}$ is given by*

$$\tilde{\Psi}_{\gamma,\Gamma} = \sum_{0 \neq P \leq P_v} \Psi_{\gamma_P} \tilde{\Psi}_{\Gamma^P}$$

The first part is proved in terms of contraction of subgraphs in proposition 3.5 of [**92**] (also as proposition 6.3 of [**8**]). We give an independent proof. The second part is more involved.

PROOF. Part (1): Let t be a spanning tree of γ and T one of Γ. Clearly $T \leftarrow t$ is a spanning tree of $\Gamma \leftarrow \gamma$ which explains the first term. Let now t be any spanning tree of $\Gamma \leftarrow \gamma$ and denote by $t|\Gamma$ the subgraph of Γ such that $t = t|\Gamma \leftarrow t|\gamma$. Clearly $H_1(t|\gamma) = 0$ but $H_1(t|\Gamma)$ may be $\neq 0$. Consequently, $\deg \tilde{\Psi}_{\gamma,\Gamma} = \deg \Psi_\gamma \Psi_\Gamma + h_1(t|\Gamma)$. (Indeed, making a tree into n homology cycles requires n additional edges.)

Part (2): Let t be a spanning tree of $\Gamma \leftarrow \gamma$ with $t = t_\Gamma \leftarrow t_\gamma$ where $t_\gamma = t|\gamma$ and $t_\Gamma = t|\Gamma$. Define a partition $P(t)$ of E_v in the following way: $e_1 \sim e_2$ for $e_1, e_2 \in E_v$ iff e_1 and e_2 are connected in t_Γ. One sees that $P(t) \leq P_v$. Using (1) we may assume that $P(t) = 0$ (the full partition/empty equivalence relation).

We show (a) that $(t_\Gamma)^{P(t)}$ is a direct spanning tree of $\Gamma^{P(t)}$ and (b) that $(t_\gamma)_{P(t)}$ is a spanning tree of $\gamma_{P(t)}$.

In order to see (a), let w_1, w_2 be two vertices of $(t_\Gamma)^{P(t)}$. Since t is a spanning tree of $\Gamma \leftarrow \gamma$ there are paths from w_1 and w_2 to some split vertices v_1, v_2 of v, respectively, and it suffices to see that v_1 and v_2 are connected in order to prove that $(t_\Gamma)^{P(t)}$ is connected. If v_1 and v_2 are in the same cell Q of $P'(t)$, then they are connected through a path in t_Γ by the very definition of $P'(t)$ which, by very definition, is $\partial b P(t)$. This fact, along with proposition 3.6 and lemma 3.4 gives us the desired result. The proof of the case when v_1 and v_2 are not in the same cell Q of $P'(t)$ is very similar– one introduces a "virtual" edge connecting v_1 and v_2 and repeats the argument. □

Note that since $\Psi_{\gamma^0} = \Psi_\gamma$ and $\tilde{\Psi}_{\Gamma_0} = \Psi_\Gamma$, we may equivalently write

$$\Psi_{\Gamma\leftarrow\gamma} = \sum_{0\le P\le P_v} \Psi_{\gamma^P} \tilde{\Psi}_{\Gamma_P}.$$

In the above description of $\Psi_{\Gamma\leftarrow\gamma}$ note the symmetry $\gamma \leftrightarrow \overline{\Gamma - v}$ in case the latter is connected. This symmetry is in fact broken by summing over partitions with respect to $\overline{\Gamma - v}$. I

3.2.3. Insertion into edges. Insertion of γ into an edge of Γ means

(i) Insert a vertex in the middle of this edge.
(ii) Insert γ into this vertex.

Having Theorem 3.7 it remains to study step (i). Let $e \in E_\Gamma$. Let Γ^e be the graph obtained from Γ by inserting a new vertex v in the middle of e. We call e_1 and e_2 the two edges of Γ^e adjacent to v.

Proposition 3.8.

$$\Psi_{\Gamma^e} = \Psi_\Gamma|_{A_e = A_1 + A_2}$$

Proof. Let t be a spanning tree of Γ. Either t includes e or it does not. If it does, $t - e \cup e_1 \cup e_2$ defines a spanning tree of Γ^e. If t does not include e, then $t - e \cup e_1$ and $t - e \cup e_2$ define spanning trees of Γ^e. Conversely, every spanning tree of Γ^e is obtained this way. □

3.3. Singularities of hypersurfaces

In this section from our previous results certain statements about the graph hypersurfaces $X_{\Gamma\leftarrow\gamma}$ themselves and their singular loci; our methods are rather elementary and depend solely on the insertion formula deduced above.

Let $\tilde{X}_{\gamma,\Gamma} = \{\tilde{\Psi}_{\gamma,\Gamma} = 0\}$. X_Γ, X_γ and $\tilde{X}_{\gamma,\Gamma}$ are projective hypersurfaces in $\mathbb{P}^{n-1}$ where $n = \#E_{\Gamma\leftarrow\gamma}$. Let CX be the affine cone over X a projective space. The question if the projective variety X_Γ or the affine variety CX_Γ is the physically relevant object to consider depends on whether Γ is divergent or not. This, in turn, depends in particular on D, the dimension of space time. In case Γ is primitive and logarithmically divergent, the projective X_Γ is the one we are interested in, as the projective period integral (1.7.17) gives the residue [**8**] of Γ. The latter is the only renormalization-scheme independent quantity that may be associated to it. In case Γ is convergent, the affine variety, and the obvious period defined on it, would be the appropriate object to look at.

3.3.1. Insertion into vertices. From part (1) of Theorem 3.7 the following statements about the cones follow immediately. The second and third one will be most useful.

Corollary 3.9.

$$\begin{aligned}
C\tilde{X}_{\gamma,\Gamma} \cap (CX_\gamma \cup CX_\Gamma) &= CX_{\Gamma\leftarrow\gamma} \cap (CX_\gamma \cup CX_\Gamma)\\
\operatorname{Sing} C\tilde{X}_{\gamma,\Gamma} \cap (CX_\gamma \cap CX_\Gamma) &= \operatorname{Sing} CX_{\Gamma\leftarrow\gamma} \cap (CX_\gamma \cap CX_\Gamma)\\
\operatorname{Sing} C\tilde{X}_{\gamma,\Gamma} \cap \operatorname{Sing} CX_\gamma &= \operatorname{Sing} CX_{\Gamma\leftarrow\gamma} \cap \operatorname{Sing} CX_\gamma\\
\operatorname{Sing} C\tilde{X}_{\gamma,\Gamma} \cap \operatorname{Sing} CX_\Gamma &= \operatorname{Sing} CX_{\Gamma\leftarrow\gamma} \cap \operatorname{Sing} CX_\Gamma
\end{aligned}$$

PROOF. Straightforward from Theorem 3.7, part (1) and the Leibniz rule. □

REMARK 3.10. The amount of information one gets out of Corollary 3.9 depends on how much of the singular locus of $\Gamma \leftarrow \gamma$ is covered by $X_\gamma \cap X_\Gamma$. There are examples where the latter intersects the singular locus in only one point (see subsection 3.4.2).

The following proposition however gives a sufficient condition for these intersections to be quite large.

PROPOSITION 3.11. *Consider $\Gamma \leftarrow \gamma$ such that P_v is the only nonempty partition subordinate to itself, Γ^{P_v} is a tree, and in γ the only two vertices that would be identified when creating γ_{P_v} are connected by one and only one edge. Then*

$$\mathrm{Sing}\, CX_{\Gamma\leftarrow\gamma} \subseteq CX_\gamma \cap CX_\Gamma \cup \mathrm{Sing}\, CX_\gamma \tag{3.3.1}$$

and therefore

$$\mathrm{Sing}\, CX_{\Gamma\leftarrow\gamma} = \mathrm{Sing}\, C\tilde{X}_{\gamma,\Gamma} \cap (CX_\gamma \cap CX_\Gamma \cup \mathrm{Sing}\, CX_\gamma) \tag{3.3.2}$$

PROOF. P_v contains only one set with two elements, write $\{e_1, e_2\}$. Let e_l be an edge joining $\partial s(e_1)$ and $\partial s(e_2)$ in γ. Note that $\Psi_{\Gamma^{P_v}} = 1$. Then, by Theorem 3.7,

$$\Psi_{\Gamma\leftarrow\gamma} = \Psi_\gamma \Psi_\Gamma + \Psi_{\gamma_{P_v}}.$$

$\Psi_{\gamma_{P_v}}$ is a multiple of A_l since e_l forms a loop in γ_{P_v}. Let $x = (A_1, \ldots, A_n) \in \mathrm{Sing}\, CX_{\Gamma\leftarrow\gamma}$. It follows that $\Psi_\gamma(x) = 0$ since Ψ_Γ is linear ($h_1(\Gamma) = 1$) and Ψ_γ, $\tilde{\Psi}_{\gamma,\Gamma}$ constant in the $e_i \in E_\Gamma$. Hence also $\psi_{\gamma,\Gamma}(x) = 0$. One concludes from $\Psi_{\gamma_{P_v}} = A_l(\Psi_\gamma|_{e_l=0})$ that both A_l and $\Psi_{\gamma_{P_v}}$ vanish. Therefore $x \in \mathrm{Sing}\, C\tilde{X}_{\gamma,\Gamma}$. By the Leibniz rule it follows that $\psi_\Gamma(x)\nabla\psi_\gamma(x) = 0$ and finally (3.3.1). In order to get from there to (3.3.2), use the second and third item of Corollary 3.9. □

A special case of the above proposition is $\Gamma = C_n$ a cycle graph and γ as above plus X_γ smooth, which means $\mathrm{Sing}\, CX_\gamma = 0 \in \mathbb{A}^{\#E_\gamma}$. In this case $\mathrm{Sing}\, CX_\gamma \subseteq \mathrm{Sing}\, C\tilde{X}_{\gamma,\Gamma}$ and (3.3.2) simplifies to

$$\mathrm{Sing}\, CX_{C_n\leftarrow\gamma} = \mathrm{Sing}\, C\tilde{X}_{\gamma,C_n} \cap CX_\gamma \cap CX_{C_n}. \tag{3.3.3}$$

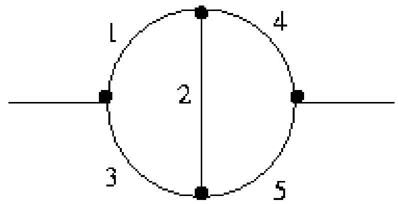
3.3.2. Insertion into edges. Recall that Γ^e denotes the graph Γ where a vertex has been inserted into the edge e. From Proposition 3.8 we know that the graph polynomial of Γ^e depends on the variables A_1 and A_2 only through their sum $A_1 + A_2$. Hence we deduce

PROPOSITION 3.12. *Let $\overline{C}X_\Gamma$ be the projective cone in $\mathbb{P}^{\#E_\Gamma}$ over the hypersurface X_Γ in $\mathbb{P}^{\#E_\Gamma - 1}$. Then*

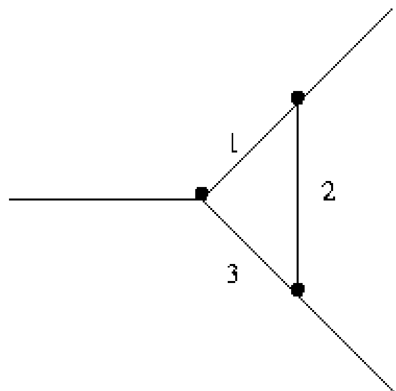
$$X_{\Gamma^e} \simeq \overline{C}X_\Gamma.$$

3.4. Examples and applications

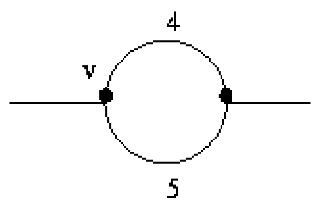
3.4.1. Two second order examples. The first example is a graph which is primitive in $D = 4$ dimensions, and therefore the period integral immediately well-defined, without renormalization of subdivergences.

$\Gamma \leftarrow \gamma =$ 

It arises from the insertion of

$\gamma =$ 

into

$\Gamma =$ 

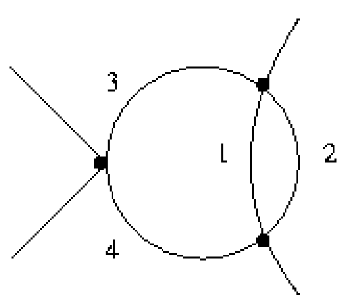
From Theorem 3.7 one obtains

$$\Psi_{\Gamma \leftarrow \gamma} = (A_1 + A_2 + A_3)(A_4 + A_5) + (A_1 + A_3)A_2,$$

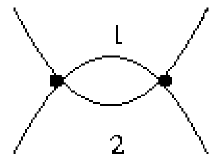
and from the remark after Proposition 3.11

$$\operatorname{Sing} CX_{\Gamma \leftarrow \gamma} = \{A_1 + A_3 = A_2 = A_4 + A_5 = 0\}.$$

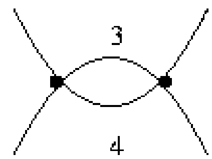
The second example is of similar nature,

$\Gamma \leftarrow \gamma =$ 

from



$\gamma =$



$\Gamma =$

One finds as above

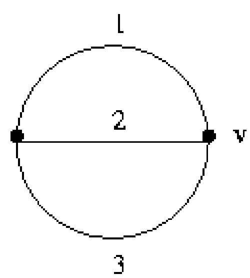
$$\Psi_{\Gamma \leftarrow \gamma} = (A_1 + A_2)(A_3 + A_4) + A_1 A_2,$$

and

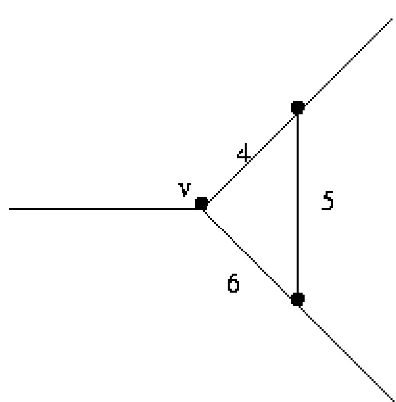
$$\operatorname{Sing} CX_{\Gamma \leftarrow \gamma} = \{A_1 = A_2 = A_3 + A_4 = 0\}.$$

$\Gamma \leftarrow \gamma$ is a non-primitive graph in $D = 4$ by powercounting, and the subdivergence needs to be subtracted in order to make the period integral well-defined.

3.4.2. The wheel with three spokes. This is an example of a graph which does not satisfy the assumptions of Proposition 3.11, and, accordingly, Corollary 3.9 can not be expected to tell anything useful.

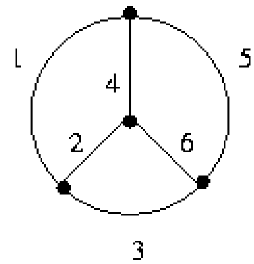


$\Gamma =$



$\gamma =$

Clearly,

$\Gamma \leftarrow \gamma =$ 

called the *wheel with three spokes*, and studied extensively in [**92, 8**].

$$\begin{aligned}\Psi_\Gamma &= A_1A_2 + A_1A_3 + A_2A_3\\ \Psi_\gamma &= A_4 + A_5 + A_6\end{aligned}$$

Applying Theorem 3.7, one easily finds

$$\tilde{\Psi}_{\gamma,\Gamma} = A_4A_5A_6 + A_1A_5(A_4 + A_6) + A_2A_6(A_4 + A_5) + A_3A_4(A_5 + A_6)$$

The first term corresponds to the partition $P_v = \{\{1,2,3\}\}$, the second to $P = \{\{1\},\{2,3\}\}$ and so on. Consequently,

$$\begin{aligned}\operatorname{Sing} C\tilde{X}_{\gamma,\Gamma} &= \{A_4 = A_5 = A_6 = 0\}\\ &\cup\{A_1 = A_2 = A_3 = A_4 = A_5 = 0\}\\ &\cup\{A_1 = A_2 = A_3 = A_4 = A_6 = 0\}\\ &\cup\{A_1 = A_2 = A_3 = A_5 = A_6 = 0\}\end{aligned}$$

and

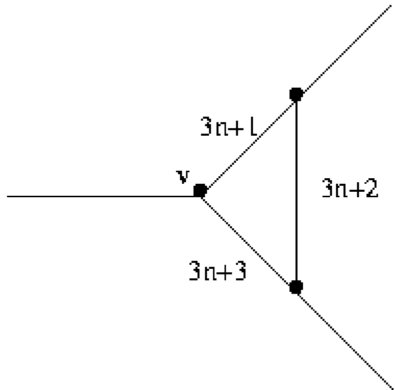
$$\operatorname{Sing} C\tilde{X}_{\gamma,\Gamma} \cap (CX_\gamma \cap CX_\Gamma) = \{A_4 = A_5 = A_6 = 0, A_1A_2 + A_1A_3 + A_2A_3 = 0\}.$$

However,

$$\begin{aligned}\operatorname{Sing} CX_{\Gamma\leftarrow\gamma} &= \{A_1A_4 + A_1A_5 + A_1A_6 + A_4A_6 = 0,\\ &\quad A_2A_4 + A_2A_5 + A_2A_6 + A_4A_5 = 0,\\ &\quad A_3A_4 + A_3A_5 + A_3A_6 + A_5A_6 = 0\}\end{aligned}$$

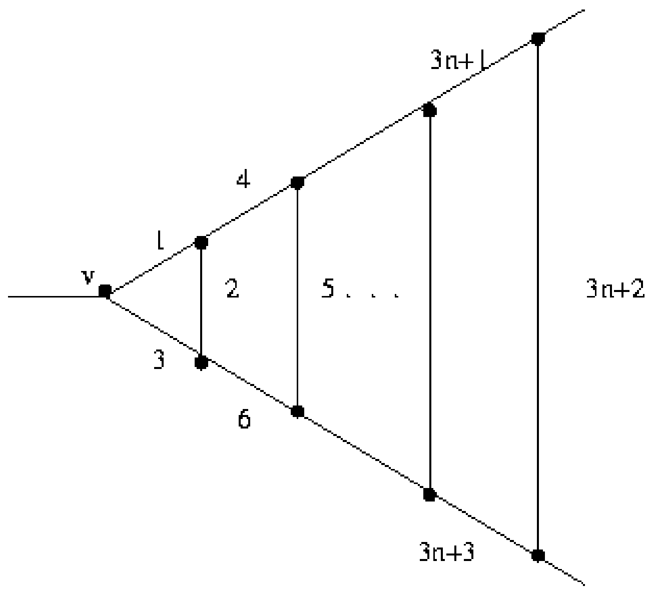
by explicit computation. Taking the intersection with $CX_\gamma \cap CX_\Gamma$ one sees only a small part of the actual singular locus. For graphs with this property one resorts to the more sophisticated methods of section 8 of [**92**].

3.4.3. Half-open ladder graphs. These form a sequence of graphs for which Proposition 3.11 does apply, and accordingly we get a number of nice results from it.

$\Gamma_n =$ 

$$\Psi_{\Gamma_n} = A_{3n+1} + A_{3n+2} + A_{3n+3}$$

We are interested in the nested insertions

$\Gamma_n \leftarrow \ldots \leftarrow \Gamma_0 =$ 

We begin with the case $n = 1$, similar to the first example in subsection 3.4.1.

$$\tilde{\Psi}_{\Gamma_0,\Gamma_1} = (A_1 + A_3)A_2$$

$$\Psi_{\Gamma_1 \leftarrow \Gamma_0} = (A_1 + A_2 + A_3)(A_4 + A_5 + A_6) + (A_1 + A_3)A_2$$

$$C\tilde{X}_{\Gamma_0,\Gamma_1} = \{(A_1 + A_3)A_2 = 0\}$$

$$\operatorname{Sing} C\tilde{X}_{\Gamma_0,\Gamma_1} = \{A_1 + A_3 = A_2 = 0\}$$

From Proposition 3.11,

$$\operatorname{Sing} C\tilde{X}_{\Gamma_1 \leftarrow \Gamma_0} = \{A_1 + A_3 = A_2 = 0, A_4 + A_5 + A_6 = 0\}.$$

Before generalizing to arbitrary n, let us look at the next step.

$$\tilde{\Psi}_{\Gamma_2,\Gamma_1 \leftarrow \Gamma_0} = ((A_1 + A_2 + A_3)(A_4 + A_6) + (A_1 + A_3)A_2)A_5$$

$$\begin{aligned}\Psi_{\Gamma_2\leftarrow(\Gamma_1\leftarrow\Gamma_0)} &= (A_1+A_2+A_3)(A_4+A_5+A_6)(A_7+A_8+A_9)\\ &\quad +(A_1+A_3)A_2(A_7+A_8+A_9)\\ &\quad +((A_1+A_2+A_3)(A_4+A_6)+(A_1+A_3)A_2)A_5\end{aligned}$$

$$C\tilde{X}_{\Gamma_2,\Gamma_1\leftarrow\Gamma_0} = \{((A_1+A_2+A_3)(A_4+A_6)+(A_1+A_3)A_2)A_5 = 0\}$$

$$\operatorname{Sing} C\tilde{X}_{\Gamma_2,\Gamma_1\leftarrow\Gamma_0} = \{(A_1+A_2+A_3)(A_4+A_6)+(A_1+A_3)A_2 = A_5 = 0\}$$

Finally, from Proposition 3.11,

$$\begin{aligned}&\operatorname{Sing} C\tilde{X}_{\Gamma_2\leftarrow\Gamma_1\leftarrow\Gamma_0}\\ = \ &\{(A_1+A_2+A_3)(A_4+A_6)+(A_1+A_3)A_2 = A_5 = A_7+A_8+A_9 = 0\}\\ \cup \ &\{A_1+A_3 = A_2 = A_4+A_6 = A_5 = 0\}.\end{aligned}$$

More generally, one finds

Proposition 3.13.

$$\tilde{\Psi}_{\Gamma_{n-1}\leftarrow\ldots\leftarrow\Gamma_0,\Gamma_n} = A_{3n-1}\left(\Psi_{\Gamma_{n-1}\leftarrow\ldots\leftarrow\Gamma_0}|_{A_{3n-1}=0}\right) \tag{3.4.1}$$

Therefore

$$CZ_{\Gamma_n\leftarrow\ldots\leftarrow\Gamma_0} = \left\{\sum_w w = 0\right\}$$

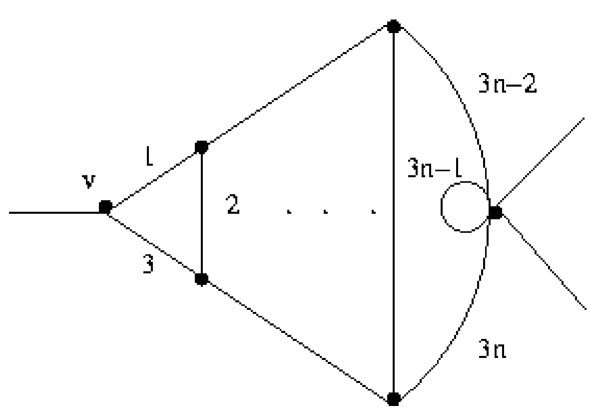
where the sum is over all words of the form $w = a_0 b_1 c_2 a_3 \ldots a_n$ *on the alphabet* $\{a_i, b_i, c_i\}$ *such that the indices are ordered from* 0 *to* n*, such that* c_0 *or* b_n *do not appear and* b_i *implies* c_{i+1}*. Finally* $a_i = A_{3i+1}+A_{3i+2}+A_{3i+3}$, $b_i = A_{3i+1}+A_{3i+3}$ *and* $c_i = A_{3i-1}$*. Furthermore,*

$$\operatorname{Sing} CX_{\Gamma_n\leftarrow\ldots\leftarrow\Gamma_0} = \left\{\sum_w \hat{w} = 0 = c_n = a_n = 0\right\}$$

where now w *are words as above ending on* c_n *and* $\hat{w}$ *is the word* w *with the last character* c_n *omitted.*

Proof. For (3.4.1) note that $P_v = \{\{3n+1, 3n+3\}, \{3n+2\}\}$, and there is no nontrivial partition strictly subordinate to it. Write $\gamma = \Gamma_{n-1} \leftarrow \ldots \leftarrow \Gamma_0$. Then γ_{P_v} is γ with the "outmost" vertices (bordering the edges $3n-2$ and $3n$) identified to a single vertex. The edge $3n-1$ becomes a loop at this new vertex which must not be part of any spanning tree of γ_{P_v}.

$\gamma_{P_v} =$



On the other hand, the tree $\{3n+1, 3n+2, 3n+3\}$ is a direct spanning tree of $\Gamma_n^{P_v}$. The second item follows from (3.4.1) in a straightforward way. For the singular locus, Proposition 3.11 is essential. □

3.5. Characteristic classes of graph hypersurfaces

Apart from studying the local geometry of the graph hypersurfaces, one would also like to obtain intersection-theoretic and topological information about them such as their Euler characteristic, for example. Unfortunately because of the fact that graph hypersurfaces are singular (with the singular loci of low codimension), our usual notion of Chern classes have to be modified in a very non-trivial way so as to be applicable to the study of singular schemes. In this section we present some examples of computation of these "modified" characteristic classes using a Macaulay 2 program written by Aluffi [**109**]and speculate how graph insertions might affect the characteristic classes thus obtained. For the first rigorous study in these direction for the case of the family of banana graphs, see [**1**].

Some background (following [**1**]): other than using the Grothendieck ring, another way of analyzing graph hypersurfaces is to use the pushforward of the *Chern-Schwarz-MacPherson* classes of the graph hypersurfaces to the Chow group. (This method does depend on the specific embedding of the graph hypersurface in the ambient projective space.) The terminology follows from the fact that MacPherson proved the existence of a theory of characteristic classes for singular varieties conjectured by Grothendieck and Deligne; independently these classes were constructed and studied by M.-H Schwarz, extending theorems of the Poincare-Hopf type to singular varieties. These CSM classes are a much more refined measure of singularities other than a simpler generalization of the notion of Chern classes to the singular case. The construction essentially depends on covariant functoriality of the assignment $X \mapsto F(X)$ where $F(X)$ is an abelian group of constructible functions and the existence of a natural transformation between the functor F and the homology functor. (See [**1**] for a nice nontechnical account.) One of the main facts about this CSM classes are that the degree of these classes equal the Euler characteristic even in the possibly singular case. CSM classes also, importantly satisfy the inclusion-exclusion principle. The Milnor class is a direct measure of the singularities of the variety in question: for example the dimension of the singular locus of a variety equals the largest dimension of a non-zero term in the Milnor class.

In [**1**] it is shown, using CSM classes, that the Euler characteristic of the hypersurfaces arising from a family of banana graphs with n parallel edges, $n \geq 3$, between two vertices is given by

$$\chi(X_{\Gamma_n}) = n + (-1)^n. \tag{3.5.1}$$

We report in the next two examples computations of the CSM classes for two interesting graphs; in all of the following H denotes the hyperplane class.

Example 3.14 (the dunce's cap (section 4.1, second example)).

Ambient space	$\mathbb{P}^3$
Ring	$\mathbb{Q}[A_1, A_2, A_3, A_4]$
Ideal	$(A_1 + A_2)(A_3 + A_4) + A_3A_4)$
Fulton class	$4H^3 + 4H^2 + 2H$
Chern-Schwartz-MacPherson class	$3H^3 + 4H^2 + 2H$
Milnor class	$-H^3$

Example 3.15 (Wheel-with-3-spokes, section 4.2).

Ambient space	$\mathbb{P}^5$
Ring	$\mathbb{Q}[A_1, A_2, A_3, A_4, A_5, A_6]$
Ideal	$(A_1A_2 + A_1 * A_3 + A_2 * A_3)A(A_4 + A_5 + A_6)+$
	$A_4A_5A_6 + A_1A_5(A_4 + A_6) + A_2A_6(A_4 + A_5) + A_3A_4A(A_5 + A_6))$
Fulton class	$27H^5 + 6H^4 + 18H^3 + 9H^2 + 3H$
Chern-Schwartz-MacPherson class	$6H^5 + 12H^4 + 14H^3 + 9H^2 + 3H$
Milnor class	$-21H^5 + 6H^4 - 4H^3$

Needless to say, both of these examples satisfy the positivity conjecture of [**1**].

In general, explicit computations of CSM classes remain very difficult; the program of [**109**] in general only works efficiently for graphs with small number of loops and edges. As a problem for the future, let us remark that one perhaps ought to study the CSM classes of a graph, before and after inserting another graph. For example consider the half-ladder graphs. As a first case, we insert one triangle into another at a fixed vertex v.

Example 3.16 (Half open ladder first insertion (6 edges)).

Ambient space	$\mathbb{P}^5$
Ring	$\mathbb{Q}[A_1, A_2, A_3, A_4, A_5, A_6]$
Ideal	$(A_1 + A_2 + A_3)(A_4 + A_5 + A_6) + (A_1 + A_3)A_2$
Fulton class	$6H^5 + 12H^4 + 14H^3 + 8H^2 + 2H$
Chern-Schwartz-MacPherson class	$5H^5 + 11H^4 + 13H^3 + 8H^2 + 2H$
Milnor class	$-H^5 - H^4 - H^3$

After another insertion of a triangle at v, we get:

Example 3.17 (Half open ladder second insertion (9 edges)).

Ambient space	$\mathbb{P}^8$
Ring	$\mathbb{Q}[A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9]$
Ideal	$(A_1 + A_2 + A_3)(A_4 + A_5 + A_6)(A_7 + A_8 + A_9)+$
	$(A_1 + A_3)A_2(A_7 + A_8 + A_9) + (A_1 + A_2 + A_3)(A_4 + A_6)A_5$
	$+(A_1 + A_3)A_2A_5$
Fulton class	$-162H^8 + 90H^7 + 54H^6 + 108H^5 + 90H^4 + 54H^3 + 18H^2 + 3H$
Chern-Schwartz-MacPherson class	$9H^8 + 34H^7 + 72H^6 + 96H^5 + 85H^4 + 50H^3 + 18H^2 + 3H$
Milnor class	$171H^8 - 56H^7 + 18H^6 - 12H^5 - 5H^4 - 4H^3$

It would be interesting to see any similarity between the general case of half-ladder graphs and the banana graphs case in (3.5.1). This will require theoretical insight to see since the program of Aluffi seems to slow down tremendously even in the case of the third insertion.

CHAPTER 4

String vacua and computation theory

4.1. Introduction

Fundamental physical theory finds itself, circa 2009, in a rather peculiar position. On one hand we have the Standard Model (SM) of particle physics with its three generations of quarks and leptons and the force mediating gauge bosons. The SM has been verified by all experiments so far with the single exception of the Higgs Boson and it is widely anticipated that it would be detected in the first runs of the LHC in the next couple of years. We also have general relativity (GR) which works remarkably well as a theory of gravity, atleast at the intermediate cosmological scales. However, when we try to unify quantum theory with GR (let alone the quantum SM with GR!) we reach a road block– for one, we do not have a *proof* whether this is *fundamentally impossible* or not. A popular claim is that a supersymmetric theory of strings and membranes in a 11-dimensional space provides *the* consistent unification of quantum field theory with GR incorporating the SM as a low energy effective theory.

The problems with this approach has been highlighted and much debated in the last couple of years (cf. [**78**]). The scientific crux of the argument against superstrings/M-theory relies on one major problem– instead of predicting a unique vacuum state which would, putatively, describe our universe, strings/M-theory admits a very large number of vacua. This collection of vacua, in analogy to problems in evolutionary biology, has been termed the string Landscape.

Much ink has been spent on merits and demerits of the string Landscape. Opponents have decried it as an example of how wrong a scientific theory can be in terms of not being falsifiable. Proponents have argued that, eventually, by studying the statistics of the Landscape– i.e. of the configuration space $\mathcal{C}$ of string vacua, $|\mathcal{C}| \sim 10^{500}$– coupled to one version of the anthropic principle or the other, we would be able to identify a unique state that describes our universe. In fact, some cosmologists (for example Vilenkin [**82**]) have argued that, along with the KKLT moduli stabilization mechanism and eternal inflation, the existence of the string Landscape points to a *multiverse* where each causally disconnected part is a universe of its own (literally!) separated from another by domain walls and each with its own set of free parameters, unification gauge group etc.

The goal of this Chapter is to examine some computation-theoretic aspects of this multiverse picture to draw the conclusion that *it may be fundamentally impossible to separate one point in the Landscape from the other*. I will do so by examining some decidability problems associated to the choice of an average unification gauge group in the multiverse. I will claim that the moduli space of metrics on a Calabi-Yau manifold has a fractal structure and argue, following the work of Nabutovsky and Weinberger, that there are several computability issues associated to the explicit construction of a Ricci-flat metric on a Calabi-Yau manifold. I will also argue that the problem of deciding whether two points on the Landscape have the same fundamental period may also be computationally intractable.

At the outset, I must mention the predecessor to this line of thought– Denef and Douglas, in an influential paper a couple of years ago, had shown that the problem of matching the observed (small) value of the cosmological constant in the Bousso-Polchinski model was NP-complete [**64**]. In the concluding sections of [**64**], they make several brief remarks which form the germ of this work.

Certain parts of this Chapter may appear quite speculative. In a paper currently under preparation [**76**], I shall furnish the full mathematical details for the arguments made here.

4.2. Decidability and gauge groups

One of the greatest discoveries in twentieth century physics has been that our material universe is best described in terms of local and global gauge symmetries. The Standard Model has, as input data, three Lie gauge groups: the abelian gauge group $U(1)$ describing electromagnetism and the nonabelian gauge groups $SU(2)$ and $SU(3)$ describing weak and strong interactions respectively. Through a phenomenom of gauge mixing between $SU(2)$ and $U(1)$, the gauge symmetry underlying the SM is

$$G_{SM} := SU(3) \times SU(2) \times U(1). \tag{4.2.1}$$

(It is customary to consider the quotient of the r.h.s of (4.2.1) by $\mathbb{Z}/6\mathbb{Z}$.) All elementary particles are then described in terms of representations of the corresponding Lie algebras.

By running the coupling constants to a sufficiently high energy ($\sim 10^{16}$ GeV) and some fine-tuning, one expects that the three fundamental forces of nature would be unified in the sense that G_{SM} would be a subgroup of a larger unification group G. There are some basic representation-theoretic restrictions on what G can or can not be based on the fact that we should at lower energies see G_{SM} of the form (4.2.1). These are (cf. [**65**]) that, for E a fixed real Lie group,

1. G should be a subgroup of E which is connected, reductive and compact and centralizing the Lorentz group $\mathrm{SL}(2, \mathbb{C})$,
2. Chirality conditions on the G_{SM}: $V_{2,1}$ is a complex representation of G ($V_{m,n}$ denotes complex representation of $G \times \mathbb{C}$), and
3. No exotic higher spin particles: $V_{m,n} = 0$ if $m + n > 4$.

Popular choices for G has been Georgi–Glashow $SU(5)$ (ruled out by proton decay experiments), Spin(10), E_6 and the Pati–Salam group $(\mathrm{Spin}(6) \times \mathrm{Spin}(4))/\mathbb{Z}/2\mathbb{Z}$.

Of course, these requirements on G are for a very distinguished point on the Landscape, namely the universe we live in! In principle, different points on the Landscape could have wildly different G. Of course, we would *like* to show that "almost all" points in the configuration space have G with a subgroup G_{SM} of the form (4.2.1) (the "naturalness condition") but diversity arguments (akin to those in evolutionary biology) forces us to consider very general Lie groups with diverse subgroups.

The central parameter for statistical analysis of gauge groups in the Landscape is the average *rank* of the gauge group (the average taken over the entire configuration space $\mathcal{C}$ with respect to a suitable measure). This average rank is then expressed in terms of the number of complex moduli of the compactified space and the configuration of D-branes wrapping it (in terms of flux).

More precisely, following [**71**][1], for X a CY 4-fold such that the orientifold limit of the F-theory compactified on X is of type IIB on the orientifold Y, the average rank of a D3-brane gauge group is [**71**]

$$\langle R_{D3} \rangle = \frac{L_*}{2n+3}, \tag{4.2.2}$$

where n is the number of complex structure moduli of Y and

$$L_* = N_{D3} + \int F^{RR} \wedge H^{NS}.$$

(N_{D3} is the net D3-brane charge.) We additionally require the tadpole cancellation condition $L_* - N_{D3} = \frac{\chi(X)}{24}$ where $\chi(X)$ is the Euler characteristic of X. In presence of a small cosmological constant Λ_*, the gauge group rank average is not significantly different from (4.2.2):

$$\langle R_{D3} \rangle_{\Lambda_*} = \frac{L_*}{2n+2}.$$

A notable fact in (4.2.2) is the absence of any parameter that depends on the explicit structure of the CY 3-fold. It is shown in [**71**] that for Y the standard orientifold $\mathbb{T}^6/\mathbb{Z}_2$ with symmetric flux, $n = 1$ and $L_* = 16$, so $\langle R_{D3} \rangle = \frac{16}{5}$ which is close to the SM gauge group rank of 4. Of course, in presence of a small c.c. for the standard orientifold, $\langle R_{D3} \rangle_{\Lambda_*} = \frac{16}{4} = 4$ which is exactly the SM gauge group rank. It is also shown that the fraction of all SUSY vacua that have gauge group rank R (of, possibly, the unification gauge group) above the SM gauge group rank R_{SM} is

$$\eta \sim \exp\Big(-\frac{R_{SM}}{\langle R \rangle}\Big) \tag{4.2.3}$$

in the large n limit. In related analysis Gmeiner et. al [**68**] estimate that the frequency of occurrence of minimally supersymmetric standard model in the Landscape (with supersymmetric intersecting D-branes on an toriodal orientifold background) is of the order 10^{-9}.

Let us fix a point in $\mathcal{C}$ and write the gauge group rank of this point as $\mathbf{r}$ and the corresponding gauge (Lie) group as $\mathbf{G}$. Let us furthermore imagine that there exists a sequence of Lie groups G_i with rank of $G_i = \alpha_i$ for $i \in I$ a finite index set such that *either* each individual G_i is a subgroup of $\mathbf{G}$ satisfying condition (1) above *or* a product of G_i is a subgroup of $\mathbf{G}$ satisfying (the rather mild!) condition (1). Other than satifying the condition (1), each G_i or their product could be any Lie group and of any real dimension; the mechanics of flux compactification does not in principle prohibit this.

The question at hand is

[1]Similar analysis was also done by Blumenhagen et. al. [**62**]

QUESTION 4.1. *Given the rank* $\mathbf{r}$ *of* $\mathbf{G}$ *and the ranks of* G_i *being* α_i*, can we find a subsequence* G_k*,* $1 \leq k < n$*,* $n = |I|$ *such that* $\alpha_1 + \alpha_2 + \ldots + \alpha_k = \mathbf{r}$*?*

Before proceeding with the answer, let us pause to understand the implication of this question. This question is another way of asking whether there is a way by which we can, by looking at rank of the (unification) gauge group of a single point in $\mathcal{C}$, determine the gauge groups at that point, after a putative symmetry breaking.

The answer is, surprisingly, that the problem is NP-complete because of the following classic theorem in computational complexity.

THEOREM 4.2 (Subset-sum problem, [**67**]). *Let* S *be a set of positive integers and* S' *a subset of* S*. Let* k *be a fixed integer. Then the decision problem of deciding whether the sum of elements of* S' *equals* k *or not is NP-complete.*

It is intuitively clear that theorem 4.2 answers question 4.1. However, a full rigorous proof has to attend to certain subtleties. Implicitly in our invoking the the subset-sum problem, we have made an assumption that the rank of a product of groups equals the sum of ranks of terms in that product. This is *prima facie* only true when the product is the free product $*$ of groups ("the Grishko–H. Neumann theorem"):

$$\mathrm{rk}(G_1 * G_2) = \mathrm{rk}\, G_1 + \mathrm{rk}\, G_2.$$

For our purposes we have the same holding true because the free product descends to the tensor product at the Lie algebra level (for Lie algebras associated to the Lie groups in question) through its universal property.

An interesting alternative to grand unification has been recently proposed by Donoghue and Pais [**66**]. In their work, instead of grand *unification*, they propose a *federation* of gauge groups by sequentially adding $SU(N)$ factors to G_{SM} for large values of N and such that the couplings converge at large enough energies. The hope then is that instead of unification, we should seek a fundamental explanation for $SU(N)$ gauge theories. At low energies, the gauge groups are "autonomous".

It is easy to see that even in this case, and perhaps in a more straight-forward way, that given a fixed gauge group rank $\mathbf{r}$, the problem of determing the "constituent" gauge groups at low energies by rank considerations is also NP-complete, like question 4.1 above. In this case for a sequence of "independent" gauge groups $G_1, G_2, \ldots, G_n$ and G_{SM} (fixed), we construct the free product

$$\Big(G_1 * G_2 * \cdots * G_n\Big) * G_{SM}$$

and basically repeat the argument as outlined above almost verbatim.

4.3. Moduli, computation and fractals

For a long time it was believed that the string Landscape– the set of string vacua with small positive cosmological constant and containing the MSSM as a low-energy effective theory– was infinite. This changed in 2006 with a paper by Acharya and Douglas [**59**]. In this work it was argued that the string Landscape may be a "discretum"; the authors argued their case on the basis of several deep "finiteness" theorems in differential geometry and topology, including some results related to the geometrization conjecture. More specifically, the gist of the Acharya–Douglas argument was that since supergravity should be a manifestation of M-theory, one ought to look at relevant finiteness results in Riemannian geometry.

One of the differential geometry results invoked was due to Cheeger which showed that, in a given sequence of smooth Riemannian manifolds M_i with volumes, diameters and sectional curvature bounded, there can only be finitely many diffeomorphism types in M_i. This helped them bound the Kaluza-Klein compactification tower and the compactification volumes. Furthermore, they also invoked a theorem due to Gromov about the space of Riemannian manifolds with fixed dimension and bounded Ricci scalar and diameter being precompact in the Gromov-Hausdorff metric (more on that soon!). Acharya and Douglas argued that such convergence conditions on the space of manifolds were needed to support a central conjecture on the Landscape:

CONJECTURE 4.3 (Acharya–Douglas [**59**], "Hypothesis 1"). *There exists a minimal distance ϵ in the configuration space $\mathcal{C}$ between physically distinct vacua.*

I shall now argue, based on computation-theoretic grounds, that the problem of determining whether there is a "minimal distance" between two points in $\mathcal{C}$ may be undecidable. In order to argue my case, let us take a detour through a fascinating world where geometry, topology and logic meet.

A central (and rather intuitive!) problem in topology is deciding whether there exists an algorithm that tells, given two smooth manifolds M and N, whether M and N are diffeomorphic (denoted as $M \simeq_{\text{diff}} N$) or not. The general strategy for answering these types of questions algorithmically is to convert the problem into a decision questions about groups (fundamental groups, homology, ...). A rather famous theorem with this flavor is due to S. Novikov.

THEOREM 4.4 (Novikov). *For all $n \geq 5$ and for all n-dimensional manifolds M and a given fixed n-dimensional manifold P, it is unsolvable whether $P \simeq_{\text{diff}} M$.*

One could ask a weaker question than that answered by theorem 4.4, namely when is a homology n-sphere (with $n > 5$ and sectional curvature bounded) diffeomorphic to the the standard n-sphere? Surprisingly enough, this problem– intimately related to the Poincaré conjecture– can be formulated as a halting problem. This was achieved by Nabutovsky and Weinberger [**74**] who showed that, for a Turing degree of unsolvability $e \in \omega$ and for every Turing machine T_e, there is a sequence of homology n-spheres $\{P^e_k\}$, $k \in \omega$ such that $P^e_k \simeq_{\text{diff}} S^n$ if and only if T_e halts on input k and *the connected sum $N^e_k = P^e_k \# M \simeq_{\text{diff}} M$ and N^e_k is associated to a local minima of the diameter functional on the space* $\mathrm{Met}(M)$ *(the space of Riemmanian metrics on M upto diffeomorphisms) with the depth of the local minima roughly equal to the settling time $\sigma_e(k)$ of the algorithm for input $y < k$* [**79**]. A slightly more formal statement is

THEOREM 4.5 ("informal thm. 0.1" of [**74**]). *For every closed smooth manifold of dimension $n > 4$, there are infinitely many local minima of the diameter functional* $\mathrm{diam} : M \to \mathbb{R}$ *on the subset* $\mathrm{Al}(M) \subset \mathrm{Met}(M)$ *of isometry classes of Riemannian metrics of curvature bounded in absolute value by 1 and the local minima is given by Riemannian metrics of smoothness $C^{1,\alpha}$ for $\alpha \in [0, 1)$. Let β be a computationally enumerable (c.e.) degree of unsolvability. Then there exists a positive constant $c(n)$ such that the local mimina of depth atleast β is β-dense* *in a path metric on* Al(M) *and the number of* β*-deep minima where the diameter is always* $\leq d$ *is no less than* $\exp(c(n)d^n)$.

Theorem 4.5 may seem rather heavy-handed, just the sort of thing that mathematicians and nobody else would care about. Let us unpack it, focussing on the space

$$\text{Met}(M) := \text{Riem}(M)/\,\text{Diff}(M), \tag{4.3.1}$$

to see its relevance for the study of string vacua[2].

First of all, we work in Met(M) as opposed to Riem(M) directly since we want to impose a Gromov-Hausdorff (GH) metric on Al(M); roughly speaking the GH metric measures distances between metric spaces and we are, after all, interested in comparing how far two Riemannian metrics are from one another. Second of all, the theorem 4.5 tells us that the infinitely many local minima of the diameter functional are metrics of smoothness $C^{1,\alpha}$. These metrics have the same properties as Riemannian structures with sectional curvature between -1 and 1 (p.6 of [**74**]) and therefore of obvious physical significance. The theorem also asserts that the number of β-deep minima is an exponential of a polynomial in the diameter bound of degree = dimension of the manifold.

In fact Nabutovsky and Weinberger prove more in [**74**]. They show that the depth of the local minima (which is roughly the settling time of the algorithm, as noted above) is in fact of the order of the "busy beaver function" in the dimension n. (The busy beaver function is an example of a function that grows faster than any computable function.) The unsolvability of the halting problem implies that *one would never be able to determine if the depth has actually been computed* (p. 7 of [**74**]). One may also ask how farther apart are these basins. It is shown in [**74**] that there are infinitely many deep basins between any two β_1- and β_2-deep minima for arbitrary Turing degrees β_1 and β_2.

We summarize the properties of the space Met(M) ([**74**], [**73**], [**72**]) in the table below. (Notation: inj is the injectivity radius, scal denotes the Ricci scalar and the Einstein-Hilbert functional is defined w.r.t the standard volume form.)

	Riemannian metrics on smooth M
Functional	$\mathrm{vol}\,/\,\mathrm{inj}^{\mathrm{n}}$
Moduli space	Riem(M)/Diff(M)
Local minima	Metrics of smoothness $C^{1,\alpha}, \alpha \in [0,1)$
Depth	$\exp(c(n)d^n)$
Einstein-Hilbert functional	$\frac{\int_M \mathrm{scal}\, d\mu_M}{\mathrm{vol}(M)^{\frac{n-2}{n}}}$
EH minimized (yes/no)	No (yes, after regularizing by adding $\epsilon\|K\|\,\mathrm{diam}^2$, $\epsilon > 0$)
Partition function	York–Gibbons–Hawking
Large scale structure	Fractal

[2]Denef and Douglas were the first to hint that the results of Nabutovsky and Weinberger could be of relevance to the Landscape. Unfortunately since the sequel of [**64**] is still awaited, we don't know what they would have done with it.

In the Calabi Yau case, we are concerned with the properties of the space

$$\mathrm{Met}_J(X) \tag{4.3.2}$$

of metrics on a Calabi-Yau (CY) manifold X with a fixed Kähler form J. Let us start with the classical definition of X. A CY manifold is a complex Kähler manifold with a trivial first Chern class and with a finite fundamental group [**83**]. The space (4.3.2) is the space of metrics on the CY manifold X. Now a natural question is what sort of metrics can be put on X? A deep theorem of Yau (settling Calabi's conjecture) is

THEOREM 4.6 (Yau, cf. [**69**]). *If X is a complex Kähler manifold with vanishing first Chern class and with Kähler form J, then there exists a unique Ricci-flat metric on X whose Kähler form J' is in the same cohomology class as J.*

The existence of these metrics are of great importance in string phenomenology– a Ricci-flat metric can be used to reduce the Einstein equation for gravity into a real Monge-Ampere equation and the solutions to the Einstein equation on a given CY can then describe the cosmological evolution of each point on the Landscape (or a universe in the multiverse.)

Unfortunately, we know very few *explicit* examples of Ricci-flat metrics on a given CY except in the most simple cases. On the other hand, we do have approximations to Ricci-flat metrics on a CY manifold to varying degrees of accuracy[3]. For example, the algebraic metrics recently studied in Headrick and Nassar [**70**] seem to approximate any *smooth* Ricci-flat metric to exponential accuracy. Furthermore, since the algebraic metrics are polynomials in the moduli parameters, several computational issues intrinsic to the solution of nonlinear PDEs such as the Einstein equations seem tractable in this approximation. The method of [**70**] is to construct an energy functional out of the Kähler volume form in such a way that minimizing this functional yields the real Monge-Ampere equation. Together with the algebraic metrics, this yields a way to solve the Einstein equations on an *algebraic* CY manifold.

However, in presence of conifold singularities, the approximation of Ricci-flat metrics by algebraic metrics seem to be not so good. Another demand made in [**70**] is that the manifolds in question should be geometrically uniform (that is, without large characteristic scales.) This raises the following question:

QUESTION 4.7. Let X be an arbitrary CY manifold (not geometrically uniform, with conifold singularities, ...). Describe the space (4.3.2).

We are interested in setting up the problem in such way as to investigate the properties of the space (4.3.2) in analogy to the space (4.3.1). Let us start with some common points of similarity between the spaces (4.3.1) and (4.3.2).

(1) Convergence in Gromov-Hausdorff topology: in the Calabi-Yau case, we have, outside a singular set, every family of Ricci-flat metrics converging to a unique singular Ricci-flat metric in the Gromov-Hausdorff topology [**77**].

[3] Of course, we have, by definition, a family of Hermitian metrics on X!

(2) Diameter bound: let (X, ω_0) be a compact n-dimensional Ricci-flat Kähler manifold and ω another Ricci-flat metric such that $\int_X \omega_0^{n-1} \wedge \omega \leq c_1$. Then the diameter of (X, ω_0) is bounded by c_1, n and ω_0 [**81**][4].
(3) The CY analogue of $\mathrm{Al}(M)$: in the Riemannian case, we are concerned with path-metrics on $\mathrm{Al}(M)$ as defined above. Essentially, this is the space of metrics with sectional curvature bounded. The analogue of this in the CY cases exists through [**81**]: it is the space of all paths $\alpha_t : [0,1] \to \overline{\mathcal{K}_{NS}}$ where $\overline{\mathcal{K}_{NS}}$ is the closure of the ample cone.

As remarked earlier, the problem of explicitly finding a Ricci-flat metric on a general Calabi-Yau is computationally very difficult. These facts coupled with the similarities between the spaces (4.3.1) and (4.3.2) suggest a conjecture similar to theorem 4.5:

CONJECTURE 4.8. *There exists a unique functional on the space (4.3.1) such that the local minima is β-deep for β a c.e. degree of unsolvability. Furthermore, the local minima are given by Ricci-flat Kähler metrics and parametrized by varying Kähler and complex moduli. The number of β-deep local minima with diameter bounds by c_1 and n (for a fixed ω_0) is an exponential of a polynomial in c_1 and n.*

Evidently, if this conjecture is true, then we would have a conceptual explanation as to why it is so hard to find an explicit Ricci-flat metric in a given Kähler class. It would, of course, also imply that the problem of exactly solving the Einstein equations on an arbitrary Calabi-Yau manifold with an explicit Ricci-flat metric is computationally and conceptually much harder than anticipated before.

4.4. Periods and string theory vacua

The last point that I'd like to bring forth about questions of algorithmic decidability and the Landscape concerns fundamental *periods* of Calabi-Yau manifolds. It is a basic claim [**61**] that the low energy effective theory with $N = 2$ supersymmetry on a Calabi-Yau 3-fold X (both the Yukawa couplings and the Kähler potential) is encoded in the periods of the manifold which is defined as

$$\varpi_i := \int_{\gamma^i} \Omega, \tag{4.4.1}$$

where Ω is a holomorphic 3-form and γ^i the basis of homology cycles of X. The fundamental period of X is ϖ_0. It is shown in [**61**] that ϖ_0 can be explicitly computed for a large class of Calabi-Yau (such as those realized as hypersurfaces in the weighed projective space or of the complete intersection type). For example, for a one parameter family of mirrors of quintic 3-folds M/G with M given by the zero locus of the polynomial $p(x, \psi) = \sum_{k=1}^{5} x_k^5 - 5\psi x_1 \cdots x_5$ and the coordinates of M identified under the action of $G = \mathbb{Z}_5^3$, it can be shown that

$$\varpi_0(\psi) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5m}} \tag{4.4.2}$$

[4]Compare this with the diameter bound in [**74**].

where $|\psi| \geq 1$ and $0 < \arg(\psi) < \frac{2\pi}{5}$. After analytic continuation to $|\psi| < 1$, we get from (4.4.2)

$$\varpi_0(\psi) = -\frac{1}{5} \sum_{m=1}^{\infty} \frac{\Gamma\left(\frac{m}{5}\right)(5\alpha^2\psi)^m}{\Gamma(m)\Gamma^4\left(1-\frac{m}{5}\right)}. \tag{4.4.3}$$

It is a theorem that for low-energy N=2 SUSY effective theory, the periods of Calabi-Yau hypersurfaces in weighed projective space with more than two moduli parameters can be expressed in terms of iterated Mellin-Barnes integrals and Horn series [**75**].

Periods, of course, are a basic arithmetic object. While periods in superstring theory are really *functions* of certain parameters (say of ψ in (4.4.3)), periods in arithmetic algebraic geometry are *numbers* of a very specific form "lying" between the algebraic closure $\overline{\mathbb{Q}}$ and $\mathbb{C}$. They are obtained from integrating an algebraic differential form over a cycle in an algebraic variety (generally defined over $\mathbb{Q}$) [**38**]. It can be verified that in the string-theoretic setting of Calabi-Yau manifolds realized as hypersurfaces in weighed projective spaces, we obtain periods in this number-theoretic sense for all values of the multiple moduli parameters.

It is, in general, a very difficult question determining whether (1) a given number is a period or not and (2) verifying whether two periods are the same or not. In the paper [**38**], Kontsevich and Zagier give several nontrivial equalities between periods. They, furthermore, conjecture that one period can be expressed as another through three basic algebraic operations– (1) additivity, (2) change of variables and (3) the Stokes formula.

One idiosyncratic view of the string Landscape (and this is decidedly mine alone!) is imagining the entire configuration space of vacua $\mathcal{C}$ as a collection of fundamental periods, one for each compactification. We can then ask whether two points on $\mathcal{C}$ are the same or not by asking whether the corresponding two fundamental periods are the same or not, for random choices of the parameter values. Notice that this is a highly relevant question for phenomenology since the merits of periods lie in their "knowledge" of the low-energy effective theory.

Now as Kontsevich and Zagier remark, in general, this question is likely to be "completely intractable now and may remain so for many years" (p.8 of [**38**]). The question is whether there is any precise way by which we can quantify this intractability. Essentially, from the computational complexity point of view, we want to get a better understanding of

(1) What sort of numbers are periods, from the computation-theoretic viewpoint?
(2) How can we distinguish periods based on their complexity such that for two given periods, it would be "easy" (= doable in polynomial time) to check whether the two are the same or not if and only if they are in the same complexity class?

Question 1 has been already answered by Yoshinaga [**84**]: in a very interesting paper, he shows that all (real) periods are computable in the sense of Turing. The proof is a mixture of facts from Tarski's quantifier-elimination theory and semi-analytic geometry of Hironaka. Let me make some brief remarks about how to tackle question 2. Let $I(s)$ be the *Igusa zeta function*

$$I(s) = \int_{\Delta_n} f^s \omega, \tag{4.4.4}$$

where s is a complex variable, $\omega = dx_1 \wedge \cdots \wedge dx_n$ and $\Delta_n \subset \mathbb{R}^{n+1}$ be the standard simplex with volume form ω. By a theorem of Belkale–Brosnan [**60**], we know that if f is a polynomial in N-variables and with $\mathbb{Q}$-coefficients and s_0 an integer, then the Laurent expansion of the Igusa zeta function

$$I(s) = \sum_{i>N} a_i (s - s_0)^i \tag{4.4.5}$$

has coefficients a_i which are periods. Now we can associate to Igusa zeta functions a canonical measure of complexity of functions, namely "heights". The idea, very roughly, for the construction of complexity classes for periods would be to compare the heights so associated to the periods a_i (cf. [**63**] for the relationship between heights and the Igusa zeta function).

Bibliography

[1] Paolo Aluffi, Matilde Marcolli *Feynman motives of banana graphs* (2008) arXiv:0801.1690v2 [hep-th]

[2] Paolo Aluffi, Matilde Marcolli *Algebro-geometric Feynman rules* (2008) arXiv:0811.2514v1 [hep-th]

[3] Paolo Aluffi, Matilde Marcolli *Parametric Feynman integrals and determinant hypersurfaces* (2009) arXiv:0901.2107v1 [math.AG]

[4] Yves André *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)* Société Mathématique de France, Paris, 2004.

[5] Prakash Belkale, Patrick Brosnan *Matroids, motives and a conjecture of Kontsevich* Duke Math. J. **116** (2003) 147–188.

[6] Franziska Bittner *The universal Euler characteristic for varieties of characteristic zero* Compos. Math. **140** 4 (2004) 1011–1032.

[7] James D. Bjorken, Sidney D. Drell *Relativistic quantum fields*, McGraw-Hill Book Co. New York, 1965.

[8] Spencer Bloch *Motives associated to graphs* Jpn. J. Math. **2** 1 (2007) 165–196.

[9] Spencer Bloch *Motives associated to sums of graphs* (2008) arXiv:0810.1313v1 [math.AG]

[10] Spencer Bloch, Arthur Ogus *Gersten conjectures and the homology of schemes* Annals. Sci. ENS **7** (1974).

[11] Spencer Bloch, Dirk Kreimer *Mixed Hodge structures and renormalization in physics* Commun.Num.Theor.Phys. **2** 4 (2008) 637–718.

[12] Christian Bogner, Stefan Weinzierl *Periods and Feynman integrals* (2007) arXiv:0711.4863v2 [hep-th]

[13] Raoul Bott *On some recent interactions between mathematics and physics* Canad. Math. Bull. **28** 2 (1985) 129–164.

[14] D.J. Broadhurst, D. Kreimer *Association of multiple zeta values with positive knots via Feynman diagrams up to* 9 *loops* Phys. Lett. B. **393** 3–4 (1997) 403–412.

[15] Alain Connes, Dirk Kreimer *Renormalization in quantum field theory and the Riemann-Hilbert problem I: The Hopf algebra structure of graphs and main theorem* Comm. Math. Phys. **210** (2000) 249–273.

[16] article Alain Connes, Dirk Kreimer *Renormalization in quantum field theory and the Riemann-Hilbert problem I: The* β*-function, diffeomorphisms and the renormalization group* Comm. Math. Phys. **216** (2001) 215–241.

[17] Alain Connes, Matilde Marcolli *Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory* Frontiers in number theory, physics, and geometry (II) Springer, Berlin (2007) 617–713.

[18] Alain Connes, Matilde Marcolli *Noncommutative geometry, quantum fields and motives* American Mathematical Society, Providence, RI, 2008.

[19] Pierre Deligne *La conjecture de Weil. I* Inst. Hautes Études Sci. Publ. Math. **43** (1974) 273–307.

[20] Pierre Deligne *Catégories tannakiennes* The Grothendieck Festschrift, Vol. II, Birkhäuser Boston (1990) 111–195.

[21] Jan Denef, Francois Loeser *Geometry of arc spaces on algebraic varieties* (2000) arXiv:0006050v1 [math.AG]

[22] Alexandru Dimca *Sheaves in topology* Springer-Verlag, New York, 2003.

[23] Dzmitry Doryn *Cohomology of graph hypersurfaces associated to certain Feynman graphs* Doctoral thesis, Universität Duisburg-Essen (2008) arXiv:0811.0402v1 [math.AG]

[24] William Fulton *Intersection theory* Springer-Verlag, New York, 1998.

[25] S.I. Gelfand, Y.I. Manin *Methods of homological algebra* Springer-Verlag, Berlin, 2003.

[26] H. Gillet. C. Soulé *Descent, motives and K-theory* J. Reine Angew. Math. **478** (1996) 127–176.

[27] D. Gieseker, H. Knörrer, E. Trubowitz *The geometry of algebraic Fermi curves* Academic Press Inc. Boston MA, 1993.

[28] D. Gieseker, H. Knörrer, E. Trubowitz *An overview of the geometry of algebraic Fermi curves* Contemp. Math. **116** Amer. Math. Soc. Providence RI. (1991) 19–46.

[29] Phillip Griffiths, Joseph Harris *Principles of algebraic geometry* John Wiley & Sons Inc. New York, 1994.

[30] Alexander Grothendieck *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5* Séminaire de Géométrie Algébrique I **1960/61** Institut des Hautes Études Scientifiques, Paris, 1963.

[31] Alexander Grothendieck *Standard conjectures on algebraic cycles*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968) Oxford Univ, London (1969) 193–199.

[32] Thomas C. Hales *What is motivic measure?* Bull. Amer. Math. Soc. (N.S.) **42** 2 (2005) 119–135.

[33] Robin Hartshorne *Algebraic geometry* Springer-Verllag, New York, 1977.

[34] Eduard Looijenga *Motivic measures* (2000) arXiv:0006220v2 [math.AG]

[35] Francois Loeser *Motivic integration and McKay correspondence* Lectures at "Trieste School and Conference on Intersection Theory and Moduli" (2002)

[36] *Motives* Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held at the University of Washington, Seattle, Washington, July 20–August 2, 1991, American Mathematical Society, Providence RI, 1994.

[37] Ludmil Katzarkov *Homological Mirror Symmetry and Algebraic Cycles* Homological Mirror Symmetry, Lecture Notes in Physics **757** Springer Verlag (2008) 1–28.

[38] Maxim Kontsevich, Don Zagier *Periods* Mathematics unlimited–2001 and beyond, Springer, Berlin (2001), 771–808.

[39] Serge Lang *Algebra* Springer-Verlag, New York, 2002.

[40] Michael Larsen *Motivic measures and stable birational geometry* Mosc. Math. J. **3** 1 (2003) 85–95.

[41] Matilde Marcolli *Motivic renormalization and singularities* (2008) arXiv:0804.4824v2 [math-ph]

[42] Matilde Marcolli *Feynman integrals and motives* Pleanry lecture at European Congress of Mathematicians 2008 (2008).

[43] Matilde Marcolli, Abhijnan Rej *Supermanifolds from Feynman graphs* J. Phys. A **41** (2008) 315402–315423.

[44] Yuri I. Manin *Correspondences, motifs and monoidal transformations* (Russian) Mat. Sb. (N.S.) **77 (119)** (1968) 475–507.

[45] Yuri I. Manin *Lectures on zeta functions and motives (according to Deninger and Kurokawa)* Astérisque **228** (1995) 121–163.

[46] Yuri I. Manin *Gauge field theory and complex geometry* Springer-Verlag, Berlin, 1997.

[47] Jacob P. Murre *Lectures on motives* Transcendental aspects of algebraic cycles, London Math. Soc. Lecture Note Ser. **313** Cambridge Univ. Press (2004) 123–170.

[48] Werner Nahm *Conformal field theory and torsion elements of the Bloch group* Frontiers in number theory, physics, and geometry. II Springer, Berlin (2007) 67–132.

[49] Chris Peters *Algebraic Fermi curves (after Gieseker, Trubowitz and Knörrer)* Séminaire Bourbaki, Vol. 1989/90, Astérisque **189–190** (1990) Exp. No. 723, 239–258.

[50] Bjorn Poonen *The Grothendieck ring of varieties is not a domain* Math. Res. Lett. **9** 4 (2002) 493–497.

[51] Jonathan Rosenberg *Algebraic K-theory and its applications* Springer-Verlag, New York, 1994.

[52] Neantro Saavedra Rivano *Catégories Tannakiennes* Lecture Notes in Mathematics **265** Springer-Verlag, Berlin, 1972.

[53] Neeraja Sahasrabudhe *Grothendieck ring of varieties* Master's Thesis, Université Bordeaux 1 (2007).

[54] A.J. Scholl *Classical motives* Motives Seattle WA 1991, Proc. Sympos. Pure Math. **55** Amer. Math. Soc., Providence, RI (1994) 163–187.

[55] Sug Woo Shin *Grothendieck function-sheaf correspondence* Lecture notes from the Harvard seminar on geometric class field theory (2005).

[56] Jan Steinstra *Motives from diffraction* Algebraic cycles and motives. Vol. 2, London Math. Soc. Lecture Note Ser. **344** Cambridge Univ. Press (2007) 340–359.

[57] Charles Weibel *The K-book: An introduction to algebraic K-theory* in preparation, chapters I–IV available at http://www.math.rutgers.edu/∼weibel/Kbook.html

[58] Don Zagier *The dilogarithm function* Frontiers in number theory, physics, and geometry. II, Springer, Berlin (2007) 3–65.

[59] Bobby Acharya, Michael Douglas *A finite Landscape?* arXiv:hep-th/0606212v1.

[60] Prakash Belkale, Patrick Brosnan *Periods and Igusa zeta functions* arXiv:math/0302090v1 [math.NT].

[61] Per Berglund, Philip Candelas, Xenia de la Ossa, Anamarie Font, Tristan Hubsch, Dubravka Jancic and Fernando Quevedo *Periods for Calabi–Yau and Landau–Ginzburg vacua* Nucl. Phys. B **419** (1994) 352–403.

[62] Ralph Blumenhagen, Florian Gmeiner, Gabriele Honecker, Dieter Lüst and Timo Weigland *The statistics of supersymmetric D-brane models* Nucl. Phys. B **713** (2005) 83–115.

[63] Antoine Chambert-Loir *Lectures on height functions* arXiv: 0812.0947v1 [math.NT].

[64] Fredrik Denef, Michael R. Douglas *Computational complexity of the landscape I* Annals Phys. **322** (2007) 1096–1142.

[65] Jacques Distler, Skip Garibaldi *There is no "Theory of Everything" inside E8* arXiv: 0905.2658v2 [math.RT].

[66] John F. Donoghue, Preema Pais *Gauge federation as an alternative to unification* Phys. Rev. D **79** (2009) 095020.

[67] Michael R. Garey, David S. Johnson *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman, 1979.

[68] Florian, Gmeiner, Ralph Blumenhagen, Gabriele Honecker, Dieter Lüst, Timo Wiegland *One in a Billion: MSSM-like D-Brane Statistics* JHEP **0601** (2006) 004.

[69] Brian R. Greene *String theory on Calabi-Yau manifolds* arXiv:hep-th/9702155v1.

[70] Matthew Headrick, Ali Nassar *Energy functionals for Calabi-Yau metrics* arXiv: 0908.2635v1 [hep-th]

[71] Jason Kumar, James D. Wells *Landscape cartography: a coarse survey of gauge group rank and stabilization of the proton* Phys. Rev. D **71** (2005) 026009.

[72] Alexander Nabutovsky *Combinatorics of the space of Riemannian structures and logic phenomena of Euclidean quantum gravity* Perspectives in Riemannian geometry, CRM Proceedings and Lecture Notes 40, Amer. Math. Soc. Providence RI (2006) 223–248

[73] Alexander Nabutovsky, Schmuel Weinberger *Critical points of Riemannian functionals and arithmetic groups* Publ. IHÉS **92** (2000) 5–62.

[74] Alexander Nabutovsky, Schmuel Weinberger *The fractal nature of Riem/Diff I* Geometriae Dedicata **101** (2003) 145–250.

[75] M. Passare, A.K. Tsikh, A.A. Cheshel *Multiple Mellin-Barnes Integrals as Periods of Calabi-Yau Manifolds With Several Moduli* Theor.Math.Phys. **109** (1997) 1544–1555.

[76] Abhijnan Rej *Algorithms and the Landscape* in preparation (2009).

[77] Wei-Dong Ruan, Yuguang Zhang *Convergence of Calabi-Yau manifolds* arXiv: 0905.3424v1 [math.DG]

[78] Lee Smolin *The trouble with physics: the rise of string theory, the fall of a science and what comes next* Penguin Books London (2008)

[79] Robert I. Soare *Computability theory and differential geometry* Bull. Symbolic Logic (2004)

[80] Max Tegmark *The mathematical universe* Found. Phys. **38** (2008) 101-150.

[81] Valentino Tosatti *Limits of Calabi-Yau metrics when the Kähler class degenerates* arXiv: 0710.4579v1 [math.DG].

[82] Alexander Vilenkin *Perspectives in cosmology* arXiv:0908.0721v1 [astro-ph.CO].

[83] Shing-Tung Yau *Calabi-Yau manifolds* Scholarpedia **4(8):6524** (2009).

[84] Masahiko Yoshinaga *Periods and elementary real numbers* arXiv:0805.0349v1 [math.AG].

[85] D. Abramovich, K. Karu, K. Matsuki, J. Wlodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc., Vol.15 (2002), N.3, 531–572.

[86] M. Aganagic, C. Vafa, *Mirror symmetry and supermanifold*, Adv. Theor. Math. Phys., Vol.8 (2004) 939–954.

[87] P. Belkale, P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. Journal, Vol.116 (2003) 147–188.

[88] C. Bergbauer, A. Rej, *Insertion of graphs and singularities of graph hypersurfaces*, preprint.

[89] I.N. Bernšteĭn, D.A. Leĭtes, *Integral forms and the Stokes formula on supermanifolds.* Functional Anal. Appl. 11 (1977), no. 1, 45–47.

[90] J. Bjorken, S. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, 1964, and *Relativistic Quantum Fields*, McGraw-Hill, 1965.

[91] S. Bloch, *Motives associated to graphs*, Japan J. Math., Vol.2 (2007) 165–196.

[92] S. Bloch, E. Esnault, D. Kreimer, *On motives associated to graph polynomials*, Commun. Math. Phys., Vol.267 (2006) 181–225.

[93] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem I. The Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys., Vol.210 (2000) 249–273.

[94] A. Connes, M. Marcolli, *Renormalization and motivic Galois theory.* International Math. Res. Notices (2004) N.76, 4073–4091.

[95] M. Kashiwara, P. Shapira, *Sheaves on manifolds*, Springer, 1990.

[96] S.P. Kumar, G. Policastro, *Strings in twistor superspace and mirror symmetry*, Phys. Lett. B, Vol.619 (2005) 163–170.

[97] M. Larsen, V. Lunts, *Motivic measures and stable birational geometry*, Moscow Math. Journal, Vol.3 (2003), N.1, 85–95.

[98] C. LeBrun, Y.S. Poon, R.O. Wells, *Projective embeddings of complex supermanifolds*, Commun. Math. Phys., Vol.126 (1990) 433–452.

[99] M. Marcolli, *Feynman motives* World Scientific (2009)

[100] Yu.I. Manin, *Gauge field theory and complex geometry*, Springer Verlag, 1988.

[101] P.B. Medvedev, *α representation in scalar supersymmetric theory*, Teoreticheskaya i matematicheskaya fizika (transl.Institute of Theoretical and Experimental Physics) Vol.35 (1978) N.1, 37–47.

[102] N. Nakanishi *Graph Theory and Feynman Integrals.* Gordon and Breach, 1971.

[103] I.B. Penkov, I.A. Skornyakov, *Projectivity and D-affinity of flag supermanifolds*, Russ. Math. Surv., Vol.40 (1987) 233–234.

[104] P. Ramond, *Field theory: a modern primer.* Addison Wesley, 1990.

[105] A.A. Rosly, A.S. Schwarz, A.A. Voronov, *Geometry of superconformal manifolds*, Comm. Math. Phys., Vol.119 (1988) 129–152.

[106] A. Schwarz, I. Shapiro, *Supergeometry and arithmetic geometry.* Nuclear Phys. B 756 (2006), N.3, 207–218.

[107] S. Sethi, *Supermanifolds, rigid manifolds and mirror symmetry.* Nuclear Phys. B 430 (1994), no. 1, 31–50.

[108] W. van Suijlekom, *The Hopf algebra of Feynman graphs in quantum electrodynamics.* Lett. Math. Phys. 77 (2006), N.3, 265–281.

[109] P. Aluffi, *Computing characteristic classes of projective schemes.* J. Symbolic Comput. 35 (2003), 1, 3–19

[110] P. Aluffi, *Characteristic classes of singular varieties* Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser (2005) 1–32

[111] C. Bergbauer, D. Kreimer, *New algebraic aspects of perturbative and non-perturbative quantum field theory* arXiv: 0704.0232 [hep-th] (2007)

[112] D.J. Broadhurst, D.Kreimer, *Knots and numbers in* ϕ^4 *theory to* 7 *loops and beyond* Internat. J. Modern Phys. C, 6, 4 (1995) 519–524

[113] D.J. Broadhurst, D.Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to* 9 *loops* Phys. Lett. B, 393, 3-4 (1995) 403–412

[114] C. Itzykson, J.B. Zuber, *Quantum field theory* McGraw-Hill International Book Co., 1980

[115] D. Kreimer, *Knots and Feynman diagrams* Cambridge University Press (2000)

[116] K. Kremnizer, M. Szczesny, *Feynman graphs, rooted trees and Ringel Hall algebras* Comm. Math. Phys., to appear

[117] R.D. MacPherson, *Chern classes for singular algebraic varieties* Ann. of Math (2), 100 (1974) 423–432

Summary

This dissertation lies at the intersection of several branches of mathematics– number theory, algebraic geometry and computation theory– with problems in modern theoretical physics, namely quantum theory, string- and M-theory, with the notion of a *period* (in the sense of Kontsevich–Zagier) as the unifying theme. Essentially it is a study of questions in and around periods and their "motivic parents" as they appear in various areas of quantum field theory and string theory.

We begin this work with a survey of certain accessible aspects of Grothendieck's theory of motives in arithmetic algebraic geometry for mathematical physicists, focusing on areas that have recently found applications in quantum field theory, especially with Feynman graphs and their motives in mind. While expository, we have intentionally emphasized areas that have either already found applications in quantum field theory or ones we see playing a larger role in the future as opposed to a general survey of motives.

We begin the purely research parts of this dissertation by generalizing the computation of Feynman integrals of log divergent graphs in terms of the Kirchhoff polynomial to the case of graphs with both fermionic and bosonic edges. This procedure gives a computation of the Feynman integrals in terms of a period on a supermanifold, for graphs admitting a basis of the first homology satisfying a condition generalizing the log divergence in this context. The analog in this setting of the graph hypersurfaces is a graph supermanifold given by the divisor of zeros and poles of the Berezinian of a matrix associated to the graph, inside a superprojective space. We introduce a Grothendieck group for supermanifolds and we identify the subgroup generated by the graph supermanifolds. This can be seen as a general procedure to construct interesting classes of supermanifolds with associated periods. This work is joint work with M. Marcolli.

From there we move to more combinatorial questions involving graph polynomials. In particular we provide a formula for the graph polynomial for a graph into which another graph has been inserted, explicitly in terms of the graph polynomials of the individual graphs. We use this formula to study several problems involving the singular loci of graph hypersurfaces. Questions such as these are deemed to be important in any theory of motivic lifts of the Connes–Kreimer Lie algebra. This Chapter is joint work with C. Bergbauer.

We end this dissertation with a speculative Chapter where it is argued that questions of algorithmic decidability, computability and complexity should play a larger role in deciding the "ultimate" theoretical description of the Landscape of string vacua. More specifically, the notion of the average rank of the (unification) gauge group in the Landscape, the explicit construction of Ricci-flat metrics on Calabi-Yau manifolds as well as the computability of fundamental periods is

examined to show that undecidability questions are far more pervasive than that described in the work of Denef and Douglas. This is seen as a way in which questions in logic and theoretical computer science will come to play a larger role in theoretical physics in the future.

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