IMSc Workshop on Noncommutative Geometry and Quantum Physics

Quantum Groups I: Mathematical Aspects

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Overview

- Hopf algebras in the language of commutative diagrams and "classical" examples,

- Quantum groups according to Drinfel'd,

- Examples of quantum groups, mostly q-deformed matrix groups,

- Elements of C^* theory of *compact* quantum groups, mainly Woronowicz's construction of $SU_q(2)$,

- Notion of isomorphism of quantum groups,

- A classification theorem for $SU_q(2)$ due to Wang and its consequence for noncommutative geometry.

Algebras, Coalgebras and Bialgebras

By an *algebra* we mean a unital, associative algebra over some field k, char $k = 0$.
An algebra A is a triple (A, μ, η) where A is a k-linear vector space,

$$\mu : A \otimes A \to A, \mu(a \otimes b) := ab$$

is the (k-linear) multiplication map and

$$\eta : k \to A,\ \text{Im}(\eta) \subset Z(A),$$

is the unit map and such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\text{id} \otimes \mu} & A \otimes A \\ \mu \otimes \text{id} \downarrow & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

and

$$\begin{array}{ccccc} A \otimes A & = & A \otimes A & = & A \otimes A \\ \text{id} \otimes \eta \uparrow & & \downarrow \mu & & \uparrow \eta \otimes \text{id} \\ A \otimes k & \xrightarrow{\sim} & A & \xleftarrow{\sim} & k \otimes A \end{array}$$

What the previous diagrams express is simply the associativity of μ and the existence of a unit in the algebra: $\eta(1_k) = 1_A$.

We say that A is commutative if $\mu = \mu \circ \tau$, where $\tau : A \otimes A \to A \otimes A$ is the flip map $(a \otimes b \mapsto b \otimes a)$.

Now one can be really adventurous and ask whether the arrows in the preceeding diagrams can be reversed. That is can we define some $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ such that they satify these "new" commutative diagrams?

The answer is yes!

Suppose we have the maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\text{id}\otimes\Delta} & A \otimes A \\ \uparrow{\scriptstyle \Delta\otimes\text{id}} & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A \end{array}$$

and

$$\begin{array}{ccccc} A \otimes A & = & A \otimes A & = & A \otimes A \\ {\scriptstyle \text{id}\otimes\epsilon}\downarrow & & \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \epsilon\otimes\text{id}} \\ A \otimes k & \xrightarrow{\sim} & A & \xleftarrow{\sim} & k \otimes A \end{array}$$

Then we have a *coalgebra*!

A coalgebra is said to be cocommutative if $\tau \circ \Delta = \Delta$.

We now demand the following compatibility conditions from Δ and ϵ:

$\Delta(ab) = \Delta(a)\Delta(b)$ and $\epsilon(ab) = \epsilon(a)\epsilon(b) \forall a, b \in A$.

With these conditions the tuple $(A, \mu, \eta, \Delta, \epsilon)$ is a *bialgebra*.

Hopf algebras

Let $(A, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Suppose $\exists S : A \to A$ such that the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta\circ\epsilon} & A \\ \Delta\downarrow & & \uparrow\mu \\ A \otimes A & \xrightarrow{\mathrm{id}\otimes S,\, S\otimes \mathrm{id}} & A \otimes A \end{array}$$

Then the tuple $(A, \mu, \eta, \Delta, \epsilon, S)$ is said to be a *Hopf algebra*. The map $S : A \to A$ is called the antipode of A.

- A cocommutative or commutative $\implies S^2 = \mathrm{id}$.

- The antipode satisfies $S(ab) = S(b)S(a)$.

- Arrow-reversal duality reverses order of composition: e.g. $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$ becomes $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$.

Classical examples of Hopf algebras

Let G be a finite group with identity e.

<u>Example 1</u>
Let kG be a vector space with basis G. It is a Hopf algebra with

- Algebra multiplication on kG to be group multiplication.

- Comultiplication on kG as $\Delta(g) = g \otimes g$.

- Unit map $\eta(1) = e$ and counit map $\epsilon(g) = 1$.

- Antipode $Sg = g^{-1}$.

This is an example of a cocommutative Hopf algebra. *If* G is commutative, then kG is commutative as well. Notice also that $S^2 = \text{id}$.

Example 2

Let $k(G) = \{f : G \to k\}$. It is a Hopf algebra with

- Algebra structure being ptwise multiplication $(fg)(x) = f(x)g(x) \forall x \in G$.

- Comultiplication being $(\Delta f)(x,y) = f(xy)$ after the identification $k(G) \otimes k(G) = k(G \times G)$.

- Unit map $\eta(1) = \text{id}$ and counit map $\epsilon(f) = f(e)$.

- Antipode $(Sf)(x) = f(x^{-1})$.

It is commutative but not cocommutative. Notice $S^2 = \text{id}$ (still).

Question: Examples of Hopf algebras with $S^2 \neq \text{id}$?

Answer:

Quantum groups!

Definition(due to Drinfel'd)
A *quantum group* is noncommutative and non-cocommutative Hopf algebra.
We qualify the adjective 'quantum' for such structures after the following:
Example
Let $k\{a, b, c, d\}$ be a free algebra and $q \in k \setminus \{0\}$. The quotient $SL_q(2) = k\{a, b, c, d\}/I_q$ is a quantum group where I_q is the ideal generated by the relations (q-commutativity)

$$ca = qac, ba = qab, db = qbd, dc = qcd, bc = cb,$$

$$da - ad = (q - q^{-1})bc$$

and the q-determinant condition

$$ad - q^{-1}bc = 1.$$

On $SL_q(2)$, the comultiplication is given by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The counit is given by

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The antipode is given by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

Remarks:

- Clearly $SL_q(2)$ is noncommutative. It is also noncocommutative which can be verified by performing $\tau\circ\Delta$. The catch (though not here!) is however to remember that an expression like $\Delta(a) = a\otimes a+b\otimes c$ is actually $\Delta(a) = a^{(1,1)} \otimes a^{(1,2)} + b^{(2,1)} \otimes c^{(2,2)}$.

- We consequently see that $S^2 \neq \mathrm{id}$ on all the generators: e.g. $S^2(b)$ and $S^2(c)$.

Thus we have a valid example of a quantum group a lá Drnifel'd. The adjective 'quantum' is justified because setting $q = 1$ in $SL_q(2)$ allows us to get $SL(2)$ with entries in some *commutative* algebra.

Conventionally,

$$q = \exp \hbar$$

so that $\hbar \to 0 \implies q \to 1$.

We note that by removing the q-det $= 1$ condition, we get $GL_q(2)$. If we choose to forget about q-det altogether, we obtain a q-deformation of the matrix group $M_q(2)$.

R.J. will talk more about q-deformed structures tommorrow. We are now ready to march towards *compact* quantum groups.

Hopf $*$- and C^*-algebras

Let (A, μ, η) be an algebra. Let $* : A \to A$ satisfy

$$* \circ * = \text{id},$$
$$* \circ \mu = \mu \circ (* \otimes *) \circ \tau.$$

The tuple $(A, \mu, \eta, *)$ is called a $*$-algebra.

The algebra $A = \mathbf{H}$ and $k = \mathbf{R}$ is a $*$-algebra where $*$ is the quaternion conjugation map. Perhaps people already see what I'm going to do next: dualize the above definition!

We want the following diagram to commute:

$$\begin{array}{ccc} A & \xrightarrow{*} & A \\ \Delta \downarrow & & \downarrow \Delta \\ A \otimes A & \xrightarrow{* \otimes *} & A \otimes A \end{array}$$

Reminder: This just states that $\Delta \circ * = (* \otimes *) \circ \Delta$, *almost* the dual of the second statement in the above definition.

Example

Recall the example of the group function Hopf algebra $k(G)$- we take $k = \mathbf{C}$ and for some $f \in \mathbf{C}(G)$, define $f^* = \bar{f}$, cpx conjugation. $\mathbf{C}(G)$ is a Hopf $*$-algebra.

Check:

We see that $\Delta(f^*)(x,y) = f^*(xy) = \overline{f(xy)} = \overline{\Delta f(x,y)} = \overline{\Sigma f^{(1)}(x) f^{(2)}(y)} = \Sigma f^{(1)*}(x) f^{(2)*}(y) = (* \otimes *)\Delta(f)(x,y)$. We have again made the identification $\mathbf{C}(G \times G) = \mathbf{C}(G) \otimes \mathbf{C}(G)$.

For the sake of completeness, we note:

Definition

A C^*-algebra is an algebra with a $*$-structure and a norm $\|.\|$, such that $*$ is compatible with the norm as $\|x^*x\| = \|x\|^2$.

We note a key embedding theorem due to Gel'fand and Nàimark that we would be using next:

Theorem

Every C^*-algebra is isomorphic to a C^*-subalgebra of bounded linear operators on a possibly ∞-dimensional Hilbert space.

We would be dealing with $SU_q(2)$ next. It is a Hopf C^*-algebra which is, to wit, the tuple $(SU_q(2), \mu, \eta, \Delta, \epsilon, *, \|.\|)$ where the morphisms will be defined shortly.

The following construction is to Woronowicz:

Let A be a $*$-algebra generated by the elements α and γ and satisfying

$$\alpha^*\alpha + \gamma^*\gamma = 1, \alpha\alpha^* + q^2\gamma^*\gamma = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*, \alpha\gamma = q\gamma\alpha, \alpha\gamma^* = q\gamma^*\alpha,$$

where q is a nonzero real number. The construction of $SU_q(2)$ goes as follows:

- Represent every element of A as an (b.l.) operator on some Hilbert space $\mathcal{H}$. To do so, it suffices to prescribe the representation $\pi : A \to B(\mathcal{H})$ only on α and γ: $\pi : \alpha \mapsto \widehat{\alpha}$ and $\pi : \gamma \mapsto \widehat{\gamma}$. Such a representation is going to be *admissible* if $\widehat{\alpha}$ and $\widehat{\gamma}$ satisy the same commutation relations as α and γ.

- For every $a \in A$, define

$$\|a\| = \sup_{\pi} \|\pi(a)\|,$$

  where the sup runs over all admissible representations. Let N be the two-sided ideal of elements of A of vanishing norm.

- Consider the quotient $\mathcal{A} := A/N$. The norm $\|.\|$ induces a norm on $\mathcal{A}$.

We define A as the completion of $\mathcal{A}$ in this norm. The C^*-algebra A is $SU_q(2)$. This is our first example of a compact quantum group.

Explicit representations of $SU_q(2)$ are hard to write down. Consult Dabrowski's "The geometry of quantum spheres".

Woronowicz gave the C^*-algebra $SU_q(2)$ the structure of a Hopf algebra as well in the following way:

Let

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

The matrix elements of u generate the C^*-algebra $\mathcal{A}$ and also A. On these elements, define the comultiplication

$$\Delta(u) = u \dot{\otimes} u,$$

and the antipode

$$S \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}.$$

The (unimportant) counit maps u to the identity matrix.

Isomorphism of $SU_q(2)$ and a classification theorem

The following definition of *isomorphism* for compact quantum groups was suggested by Wang. Here we will state it only for $SU_q(2)$. Let $SU_q(2)$ and $SU_{q'}(2)$ have comultiplication Δ and Δ' respectively. Let

$$\Gamma : SU_q(2) \to SU_{q'}(2)$$

be a $*$-homomorphism. That is, $\Gamma(x^*) = \Gamma(x)^*$. We demand of Γ that the it makes the following diagram commute:

$$\begin{array}{ccc} SU_q(2) \otimes SU_q(2) & \xrightarrow{\Gamma\otimes\Gamma} & SU_{q'}(2) \otimes SU_{q'}(2) \\ \Delta\Big\uparrow & & \Big\uparrow\Delta' \\ SU_q(2) & \xrightarrow{\Gamma} & SU_{q'}(2) \end{array}$$

If Γ is an isomorphism, then we say that $SU_q(2)$ are $SU_{q'}(2)$ are isomorphic.

A theorem of Wang states that:

For q, q' nonzero reals, $SU_q(2)$ and $SU_{q'}(2)$ are isomorphic in the preceeding sense (that is, as Hopf C^*-algebras) if and only if $q' = q$ or $q' = q^{-1}$.

Contrast this with a theorem of Woronowicz which states that $SU_q(2)$ and $SU_{q'}(2)$ are always isomorphic as C^*-algebras.

These two theorems, taken together, have a rather interesting consequence for noncommutative geometry!

Recall the first Gel'fand-Naimark theorem: Given a (compact, Hausdorff) space X and a commutative C^*-algebra of functionals on X with unit $C(X)$, we can "recover" all the properties of X from $C(X)$. In particular, if X and Y are homeomorphic, then $C(X)$ and $C(Y)$ are isomorphic as C^*-algebras.

What Woronowicz's theorem is stating is that, IF we are to imagine $SU_q(2)$ as a C^*-algebra of functionals on some "space" X_q, then this correspondence $X_q \to C(X_q) := SU_q(2)$ is "insensitive" to the choice of the deformation parameter q. Making $q \to q'$ will still give us the same C^*-algebras.

On the other hand, what Wang's theorem is suggesting is that IF we require that the correspondence $X_q \to C(X_q)$ give us a Hopf C^*-algebra of functionals on the same "space", then the correspondence is sensitive to the choice of q. Scaling q will (almost always) give us nonisomorphic Hopf C^*-algebras.