C^*-Algebra Approach to Quantum $SU(2)$ Groups

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PREFACE

The goal of this thesis is threefold. In the first chapter, we present a fairly comprehensive introduction to the theory of quantum groups (noncommutative and noncocommutative Hopf algebras) in general categorical terms. We then review *-stuctures on Hopf algebras and discuss the C^*-algebra approach to the quantum group $SU_q(2)$ which can be viewed as a deformation of the classical matrix group $SU(2)$. Our discussion follows the approach of Woronowicz. In this chapter, we also discuss the basic ideas behind noncommutative/quantum spaces. In the last chapter, we review a certain isomorphism theorem for $SU_q(2)$ and see the surprising fact that by varying the deformation parameter q, we obtain nonisomorphic $SU_q(2)$-algebras.

Chapter 1

Introduction to Hopf algebras and quantum groups

1.1 Generalities

The main goal of this chapter is two-fold:

1. We introduce a basic construction of matrix groups from free associative algebras as a warm-up to questions about q-deforming such structures.

2. We construct certain Hopf algebras and their q-deformations, specifically examining quantum matrix groups.

An auxillary goal of this chapter is to fix the notation to be used in the rest of the thesis.

Throughout, we work over a field k. However, as we go along we will be imposing additional structure on k (fixing its characteristic, its algebraic closure as well as specifying a topology). Tensor products are understood to be taken over this field.

Almost all of the definitions and constructions here are from Majid [8] and Kassel [6] and no claim to originality is made with respect to either.

1.2 Matrix groups from free associative algebras

We start with the definition of a k-algebra. (Unless specified, "algebra" will be always taken to mean "associative algebra with unit".)

Definition 1.1. *Let A be a k-vector space. A is a k-algebra, if in addition, we have:*

(multiplication) *an associative k-linear map $\mu_A : A \otimes A \to A$,*

(unit) *a k-linear multiplicative map $\eta_A : k \to A$, $\eta_A(1) = 1_A$ where $1 \in k$ is the unit of k and $1_A \in A$ is the unit of A.*

We occasionally denote $\mu_A(a \otimes b)$ as ab.

Remark 1.2. Definition 1.1 can be recast in the following way: A triple (A, μ_A, η_A) is a k-algebra where the following diagram commute (associativity of the multiplication map μ_A):

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \mu_A} & A \otimes A \\ \Big\downarrow{\scriptstyle \mu_A \otimes \mathrm{id}_A} & & \Big\downarrow{\scriptstyle \mu_A} \\ A \otimes A & \xrightarrow{\mu_A} & A \end{array}$$

where $\mathrm{id}_A : A \to A$ is the identity map.

The requirement on the unit map η_A is that the following diagram commutes:

$$\begin{array}{ccccc} A \otimes A & = & A \otimes A & = & A \otimes A \\ \Big\uparrow{\scriptstyle \mathrm{id}_A \otimes \eta_A} & & \Big\downarrow{\scriptstyle \mu_A} & & \Big\uparrow{\scriptstyle \eta_A \otimes \mathrm{id}_A} \\ A \otimes k & \xrightarrow{\sim} & A & \xleftarrow{\sim} & k \otimes A \end{array}$$

Given a k-algebra A, we define the opposite k-algebra A^{op} as a k-algebra with the multiplication

$$\mu_{A^{op}} = \mu_A \circ \tau,$$

where τ is the *flip* automorphism

$$\tau : A \otimes A \to A \otimes A,$$

$$a \otimes b \mapsto b \otimes a.$$

Definition 1.3. *If $\mu_{A^{op}} = \mu_A$, we say that the k-algebra A is commutative.*

Remark 1.4. We can express the definition of a commutative k-algebra in the language of commutative diagrams: we say A is a commutative k-algebra if the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_A} \\ A & = & A \end{array}$$

One constructs the category of k-algebras (in the sense of Definition 1.1) with the obvious objects (k-algebras) and with ring homomorphisms $f : A \to B$ as morphisms where A and B are k-algebras and with the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \eta_A} & & \uparrow{\scriptstyle \eta_B} \\ k & = & k \end{array}$$

We refer to this category as **k-Alg** and denote the set of (algebra) morphisms from A to B as $\mathrm{Hom}_{\text{k-Alg}}(A, B)$. It should be noted that elements of $\mathrm{Hom}_{\text{k-Alg}}(A, B)$ preserve units.

If $i : A \to B$ is an *injective* algebra morphism, we say that A is a *subalgebra* of B.

We now give a definition of *free* k-algebra.

Definition 1.5. *Let X be a set. Consider the vector space $k\{X\}$ with the basis the set of all words $x_{i_i} \cdots x_{i_k}$ in the alphabet X (including the empty word $\emptyset$). Concatenation of words define the multiplication on $k\{X\}$:*

$$(x_{i_i} \cdots x_{i_p}) \cdot (x_{i_{p+1}} \cdots x_{i_n}) = x_{i_i} \cdots x_{i_p} x_{i_{p+1}} \cdots x_{i_n} \in k\{X\}.$$

Furthermore, we regard $\emptyset$ as the unit of $k\{X\}$. With this structure, $k\{X\}$ is called a free k-algebra.

Remark 1.6. Every object in **k-Alg** can be obtained by taking the quotient of $k\{X\}$ for a suitable X by a suitable two-sided ideal.

From here and on, we will be concerned with the free k-algebra on the finite set $X = \{x_1, \ldots, x_n\}$. We note:

Proposition 1.7. *The free polynomial algebra $k[x_1, \ldots, x_n] \cong k\{x_1, \ldots x_n\}/\mathcal{I}$ where $\mathcal{I}$ is a (two-sided) ideal of $k\{x_1, \ldots x_n\}$ generated by elements of the form $x_i x_j - x_j x_i, 1 \leq i, j \leq n$.*

From now on, we will be mainly concerned with polynomial algebras over a set of four variables $\{a, b, c, d\}$ (our eventual goal being to study deformations of two-dimensional matrix algebras). In this case, the generators of the ideal $\mathcal{I}$ can be written down explicitly as $ab - ba, bc - cb, ac - ca, db - bd, dc - cd, da - ad$.

Remark 1.8. By slightly abusing the terminology, we refer to $k[a, b, c, d]$ as an *affine space*, and given a commutative k-algebra A, elements of $\mathrm{Hom}_{\text{k-Alg}}(k[a, b, c, d], A)$

as *A-points of the affine space*. We will need to remember this when we deal with more geometric questions in the latter chapters.

Denote $k[a,b,c,d]$ by $M(2)$. Let A be a commutative k-algebra and let $M_2(A)$ be the algebra of 2×2 matrices with entries in A.

Proposition 1.9. *$Hom_{k\text{-}Alg}(M(2), A) \cong M_2(A)$ as k-vector spaces.*

Proof. (Outline) We use the fact that $M_2(A)$ is in bijection with A^4 of 4-tuples as sets. Define $\phi : \mathrm{Hom}_{\text{k-Alg}}(M(2), A) \to M_2(A)$ as

$$\phi(f) = \begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}$$

where $f \in \mathrm{Hom}_{\text{k-Alg}}(M(2), A)$. The map ϕ is a bijection. □

Make the identification $M(2) \otimes M(2) = k[a', a'', b', b'', c', c'', d', d'']$.

Proposition 1.10. *Let $\Delta : M(2) \to M(2) \otimes M(2)$ be an algebra morphism defined on the generators of $M(2)$ by*

$$\Delta(a) = a' \otimes a'' + b' \otimes c'', \Delta(b) = a' \otimes b'' + b' \otimes d'',$$

$$\Delta(c) = c' \otimes a'' + d' \otimes c'', \Delta(d) = c' \otimes b'' + d' \otimes d''.$$

For any commutative k-algebra A, Δ corresponds to matrix multiplication in $M_2(A)$.

Proof. Introduce the matrix notation

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tilde{A}' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \tilde{A}'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

We see that

$$\Delta(\tilde{A}) = \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \tilde{A}'\dot{\otimes}\tilde{A}'',$$

where $\tilde{A}'\dot{\otimes}\tilde{A}''$ denotes the usual product of matrices $\tilde{A}'$ and $\tilde{A}''$ where the product between matrix elements is replaced by the tensor product. □

1.3 Co-, bi- and Hopf algebras

Let C be a k-vector space with unit 1_C.

Definition 1.11. *A k-*coalgebra *C is the triple $(C, \Delta_C, \epsilon_C)$ where*

(comultiplication) $\Delta_C : C \to C \otimes C$ *is a k-linear map satisfying the commutative diagram*

$$\begin{array}{ccc} C \otimes C & \xrightarrow{\Delta_C \otimes id_C} & C \otimes C \otimes C \\ \uparrow{\scriptstyle \Delta_C} & & \uparrow{\scriptstyle id_C \otimes \Delta_C} \\ C & \xrightarrow{\Delta_C} & C \otimes C \end{array}$$

(counit) $\epsilon_C : C \to k$ *is a k-linear map satisfying the commutative diagram*

$$\begin{array}{ccccc} C \otimes C & = & C \otimes C & = & C \otimes C \\ \downarrow{\scriptstyle \epsilon_C \otimes id_C} & & \uparrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle id_C \otimes \epsilon_C} \\ k \otimes C & \xrightarrow{\sim} & C & \xleftarrow{\sim} & C \otimes k \end{array}$$

Remark 1.12. One notices that the definition of a coalgebra is obtained simply by reversing the arrows in the commutative diagrams that express the definition of a k-algebra (see remark 1.2). At the first pass, this may look like a purely categorical trick devoid of any explicit meaning. However, one finds a number of examples of coalgebras that can be constructed quite explicitly and resembling familiar constructions. A good example of this is given in proposition 1.10.

A cocommutative k-algebra is obtained by reversing the arrows in remark 1.4.

Example 1.13. *The ground field k has a natural coalgebra structure by setting $\Delta_k(1) = 1 \otimes 1$ and $\epsilon_k(1) = 1$.*

The category of k-coalgebras **k-coAlg** has k-coalgebras as objects and maps $h : C \to D$ as morphisms where C and D are k-coalgebras and such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle \epsilon_C} & & \downarrow{\scriptstyle \epsilon_D} \\ k & = & k \end{array}$$

We now define a k-*bialgebra*. The basic idea is this: a k-bialgebra is a k-algebra *and* a k-coalgebra with a *compatibility condition* between morphisms in the corresponding categories **k-Alg** and **k-coAlg**. More precisely:

Definition 1.14. *Let B be a k-algebra and a k-coalgebra with multiplication and comultiplication maps μ_B and Δ_B respectively and unit and counit maps η_B and ϵ_B respectively. If Δ_B and ϵ_B are algebra morphisms, then we say that B is a* k-bialgebra.

Remark 1.15. The compatibility condition Δ_B and ϵ_B being algebra morphisms is equivalent to the condition that μ_B and η_B are coalgebra morphisms. $B \otimes B$ has a natural algebra structure given by component-wise operations.

From now on, we simplify our notation: given a bialgebra B, we write Δ

and μ for the comultiplication and multiplication maps and η and ϵ for the unit and counit maps respectively. We will drop the subscript B.

Recall that if Δ and ϵ are algebra morphisms, then

$$\Delta(xy) = \Delta(x)\Delta(y)$$

and

$$\epsilon(xy) = \epsilon(x)\epsilon(y).$$

Furthermore, we require $\Delta(1_B) = 1_B \otimes 1_B$ and $\epsilon(1_B) = 1_k$ where 1_k is the identity on k.

Keeping the notation of proposition 1.10, we have:

Example 1.16. *M(2) is a k-bialgebra.*

Remark 1.17. We note that the identity map $\mathrm{id} : B \to B = \mu \circ \Delta$, where B is a k-bialgebra.

Given a k-bialgebra H and $f, g \in \mathrm{Hom}_{\text{k-Alg}}(H, H)$, we have the sequence

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

Define the *convolution* on $\mathrm{Hom}_{\text{k-Alg}}(H, H)$ as

$$f \bullet g = \mu \circ (f \otimes g) \circ \Delta.$$

Let $S \in \mathrm{Hom}_{\text{k-Alg}}(H, H)$ such that

$$S \bullet \mathrm{id} = \mathrm{id} \bullet S = \underline{1},$$

where $\underline{1} = \eta \circ \epsilon$. The map S is called the *antipode*. We assume that S exists and is unique. (The proof of the uniqueness of S, stripped of notational complexity, is analogous to showing that inverses of elements of a group are unique, cf. Majid [8].)

Definition 1.18. *The tuple* $(H, \mu, \eta, \Delta, \epsilon, S)$ *is called a* Hopf algebra*.*

Alternatively, given a k-bialgebra H and an antipode S, we call H a Hopf algebra.

Remark 1.19. An alternative way of looking at S is by requiring $S : H \to H$ be a map such that the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\eta\circ\epsilon=\underline{1}} & H \\ \downarrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \mu} \\ H \otimes H & \xrightarrow{S\otimes \mathrm{id}, \mathrm{id}\otimes S} & H \otimes H \end{array}$$

In the next two examples, let G be a finite group with identity e.

Example 1.20. (group Hopf algebra) *Let kG be a k-vector space with basis G. Define*

- *(algebra) multiplication on kG to be the group multiplication,*
- *comultiplication on kG $\Delta(g) = g \otimes g$,*
- *counit in kG $\epsilon(g) = 1$, and*
- *antipode $Sg = g^{-1}$,*

where $g \in G$. With these operations, kG is a Hopf algebra with Δ and ϵ extended to all of kG by linearity.

Proof. The algebra structure of kG directly comes from the group structure. Therefore, we only verify that Δ, ϵ and S satisfy the required axioms. $\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h)$. Similarly $\epsilon(gh) = \epsilon(g)\epsilon(h)$. So Δ and ϵ are algebra morphisms. We check that S satisfies the antipode definition:

$$(\mu \circ (S \otimes \mathrm{id}) \circ \Delta)g = (\mu \circ (S \otimes \mathrm{id}))g \otimes g = \mu(g^{-1} \otimes g) = e.$$

Similarly

$$(\mu \circ (\mathrm{id} \otimes S) \circ \Delta)g = (\mu \circ (\mathrm{id} \otimes S))g \otimes g = \mu(g \otimes g^{-1}) = e.$$

Therefore, S is verified to be the antipode. □

Example 1.21. (group function Hopf algebra) *Let $k(G) = \{f : G \to k\}$. With respect to pointwise multiplication $(fg)(x) = f(x)g(x)$ for all $x \in k(G)$, $k(G)$ is a k-algebra . Identify $k(G) \otimes k(G) = k(G \times G)$ and define*

$$(\Delta f)(x, y) = f(xy), \ \epsilon(f) = f(e) \text{ and } (Sf)(x) = f(x^{-1}) \text{ for all } x \in G.$$

Then $k(G)$ is a Hopf algebra.

Remark 1.22. The identification $k(G) \otimes k(G) = k(G \times G)$ does not hold true if G is infinite.

Proof. (of 1.21) The algebra structure of $k(G)$ is obvious. Therefore we simply verify that Δ, ϵ and S satisfies the required axioms.

$$\Delta f(x,y)\Delta g(x,y) = f(xy)g(xy) = (fg)(xy) = (\Delta fg)(x,y).$$

We verify that Δ is coassociative:

$$(\Delta \circ (\Delta \otimes \mathrm{id})f)(x,y,z) = \Delta f(xy,z)$$

$$= f((xy)z) = f(x(yz)) = \Delta f(x,yz) = (\Delta \circ (\mathrm{id} \otimes \Delta))f(x,y,z).$$

A similar argument checks that ϵ is also an algebra morphism. □

Remark 1.23. We note that the coassociativity in the above example comes from the associativity of group multiplication.

1.4 Quantum groups

With the machinery developed in the previous sections, we (finally) come to a very basic definition:

Definition 1.24. (Drinfel'd [5]) *A quantum group is a noncommutative and non-cocommutative Hopf algebra.*

Let $q \in k \setminus \{0\}$.

Example 1.25. *The quotient algebra* $SL_q(2) = k\{a,b,c,d\}/_{\mathcal{I}_q}$ *is a quantum group where* $\mathcal{I}_q$ *is the ideal generated by the six* q-commutativity relations

$$ca = qac, ba = qab, db = qbd, dc = qcd, bc = cb, da - ad = (q - q^{-1})bc$$

and the q-determinant relation

$$ad - q^{-1}bc = 1.$$

The comultiplication is given by

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The counit is given by

$$\epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The antipode is given by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

Proof. Freeness implies that $\Delta(ab) = \Delta(a)\Delta(b)$, so it suffices to check that $\Delta(ab)$ is well defined:

$$\Delta(ab) = \Delta(a)\Delta(b) = (a \otimes a + b \otimes c)(a \otimes b + b \otimes d)$$

$$= a^2 \otimes ab + ab \otimes ad + ba \otimes cb + b^2 \otimes cd \in SL_q(2) \otimes SL_q(2).$$

Similar checks can be performed for the other generators as well. The well-definedness of ϵ is checked in a similar way.

To see that S satisfies the antipode axioms,

$$(\mu \circ (S \otimes \mathrm{id}) \circ \Delta)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (\mu \circ (S \otimes \mathrm{id})) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \mu \left(\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

$$= \begin{pmatrix} d \otimes a - qb \otimes c & d \otimes b - qb \otimes d \\ -q^{-1}c \otimes a + a \otimes c & -q^{-1}c \otimes b + a \otimes d \end{pmatrix}$$

$$= \begin{pmatrix} a \otimes d - q^{-1}b \otimes c & 0 \\ 0 & a \otimes d - q^{-1}b \otimes c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also have

$$(\mu \circ (\mathrm{id} \otimes S) \circ \Delta) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (\mu \circ (\mathrm{id} \otimes S)) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \right)$$

$$= \begin{pmatrix} a \otimes d - q^{-1}b \otimes c & -qa \otimes b + b \otimes a \\ c \otimes d - q^{-1}d \otimes c & -qc \otimes b + d \otimes a \end{pmatrix}$$

$$= \begin{pmatrix} a \otimes d - q^{-1}b \otimes c & 0 \\ 0 & a \otimes d - q^{-1}b \otimes c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, S satisfies the antipode axioms. $\square$

Remark 1.26. If we drop the q-determinant condition in the previous example and keep the rest of the structures, we obtain the quantum analog of the group of 2×2 matrices $M_q(2)$.

It is obvious from the previous example that if we set $q = 1$, we recover the classical matrix group $SL(2)$. However, the role of q in the theory of quantum groups is deeper than being a mere fudge factor in classical constructions. At this point, we delve further into the meaning of q and justify the label *quantum* for noncommutative and noncocommutative Hopf algebras.

1.5 Deformation and quantization

The original motivation for studying quantum groups come from physics, specifically from the study of quantum integrable systems and inverse scattering (the work of Fadde'ev and others) on one hand and understanding *quantization* as *deformation* of some pre-existing structure, on the other. In this section, we focus on the latter to get a better understanding of the meaning of q by examining the latter motivation.

Roughly, in order for one to do physics, one needs *states* and *observables*; typically, in classical mechanics, the states are points in some $2n$-dimensional manifold (the phase space) and observables are $\mathbf{R}$-valued functions on this manifold. Now the set of observables in classical mechanics $\mathcal{O}_{CM}$ can be readily given the structure of an $\mathbf{R}$-algebra with respect to (pointwise) multiplication and addition of functions. The obvious but crucial feature of $\mathcal{O}_{CM}$ is that it is commutative. On the other hand, states in quantum mechanics correspond to points in some (possibly infinite dimensional) complex Hilbert space and observables in quantum mechanics correspond to Hermitian operators on this Hilbert space. The set of all observables in quantum mechanics $\mathcal{O}_{QM}$ forms a $\mathbf{C}$-algebra that is no longer commutative: this is charaterized by the Heisenberg relation

$$[\hat{x}, \hat{p}] = i\hbar,$$

where $\hbar$ is a constant and $[\hat{x}, \hat{p}]$ the commutator of $\hat{p}$ and $\hat{x}$, the position and momentum operators respectively. (Canonical) quantization, then, is the prescription

$$p \mapsto \hat{p} \text{ and } x \mapsto \hat{x},$$

where p and x are classical momentum and position and such that $\hat{p}$ and $\hat{x}$ satisfy the Heisenberg relation.

One sees that if $\hbar = 0$, then the classical and quantum descriptions coincide in the sense that $\mathcal{O}_{QM}$ becomes commutative. This was the motivation of Moyal [10] when he introduced a new product $\star_\hbar$ on $\mathcal{O}_{CM}$ such that

$$\lim_{\hbar \to 0} \frac{x \star_\hbar p - p \star_\hbar x}{\hbar} = \{x, p\},$$

where $\{x, p\}$ is the Poisson bracket on the classical phase space (ie, a $2n$-dimensional manifold with a symplectic structure given by the Poisson bracket.) Equipped with the $\star_\hbar$ product, $\mathcal{O}_{CM}$ is noncommutative and contains classical mechanics as a degenerate case.

The relation between quantum groups (say $SL_q(2)$) and the classical group (say $SL(2)$) is analogous to the relation between $\mathcal{O}_{QM}$ and $\mathcal{O}_{CM}$ respectively. It is a convention to view the deformation parameter $q = \exp \hbar$. (Then, $\hbar \to 0 \implies q \to 1$.) The quantum case includes the classical case as a special case and comes with a richer structure than the classical degenerate counterpart. Furthermore, quantum groups are symmetrical under the arrow-reversal duality (i.e, they are Hopf algebras) while classical groups are not.

Chapter 2

Hopf $*$-algebras and the philosophy of Woronowicz and Connes

2.1 Generalities

In this Chapter, we introduce a $*$-structure (involution) on a Hopf algebra and define a Hopf $*$-algebra. As a principal example, we present $SU_q(2)$, the quantum analog of the classical group of special unitary matrices $SU(2)$. The latter part of the chapter is devoted to situating the study of quantum groups in the natural geometric context developed by Connes in his noncommutative geometry program. To this end, we present a discussion of the classical Gelfand duality in a categorical language and use it as a foundation for discussing the philosophy of Connes and Woronowicz who first developed the study of C^*-algebras of quantum groups.

2.2 Hopf *-algebras

As usual, let k be a field. For simplicity, assume char $k = 0$; the reader may safely assume k to be $\mathbf{R}$ or $\mathbf{C}$.

Let A be a k-algebra with multiplication μ and unit η. Let $* : A \to A$ satisfy

$$* \circ * = \text{id}, \text{the identity map},$$

$$* \circ \mu = \mu \circ (* \otimes *) \circ \tau,$$

where $\tau : A \otimes A \to A \otimes A$ is the flip automorphism. With this map, A is called a $*$-algebra. Let A and B be $*$-algebras. Let $\phi \in \text{Hom}_{\text{k-Alg}}(A, B)$. The map ϕ is a $*$-morphism if $\phi \circ * = * \circ \phi$, that is, ϕ and $*$ commute under composition of maps.

Let $k = \mathbf{C}$ and $A = \mathbf{H}$, the ring of quaternions. Define $* : \mathbf{H} \to \mathbf{H}$ by $h \mapsto h^* = \bar{h}$, that is, $*$ is the quaternion conjugation map. One readily verifies that $\mathbf{H}$ is a $*$-algebra. We also see that with $k = \mathbf{R}$, $\mathbf{C}$ is also a *-algebra that is commutative.

Let H be a k-Hopf algebra with identity id_H, to wit, the tuple $(H, \mu, \eta, \Delta, \epsilon, S)$.

Definition 2.1. *H is a Hopf $*$-algebra if H is also a $*$-algebra and the following diagrams commute:*

$$\begin{array}{ccc} H & \xrightarrow{\;*\;} & H \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta} \\ H \otimes H & \xrightarrow{\;*\otimes*\;} & H \otimes H \end{array}$$

$$\begin{array}{ccc} H & = & H \\ \Big\downarrow{\scriptstyle *} & & \Big\uparrow{\scriptstyle *} \\ H & \xrightarrow{\;S\;} & H \end{array}$$

and

$$\begin{array}{ccc} k & = & k \\ \Big\uparrow{\scriptstyle \epsilon} & & \Big\uparrow{\scriptstyle \epsilon} \\ H & \xrightarrow{\;*\;} & H \end{array}$$

Example 2.2. *Recall example 1.21. Let $k = \mathbf{C}$. The group function Hopf algebra $\mathbf{C}(G)$ has a natural $*$-structure: For $f \in \mathbf{C}(G)$, define $f^* = \bar{f}$, the complex conjugate of f (obtained by taking the complex conjugate of the $\mathbf{C}$-coefficients of f.) With this structure, $\mathbf{C}(G)$ is a Hopf $*$-algebra.*

Proof. We see that $\Delta(f^*)(x,y) = f^*(xy) = \overline{f(xy)} = \overline{\Delta f(x,y)} = \overline{\sum f_1(x) f_2(y)} = \sum {f_1}^*(x) {f_1}^*(y) = (* \otimes *)\Delta(f)(x,y)$. In claiming $\Delta f(x,y) = \sum f_1(x) f_2(y)$, we have used the identification $\mathbf{C}(G \times G) = \mathbf{C}(G) \otimes \mathbf{C}(G)$. □

For the sake of completeness, we note:

Definition 2.3. *A C^*-algebra is a $\mathbf{C}$-algebra with a $*$-structure with a norm $\| \cdot \|$ compatible with the $*$-structure: $\|x^* x\| = \|x^2\|$.*

We also note a key embedding theorem about C^*-algebras that we will be using a few times in the next section.

Theorem 2.4. (Gelfand–Naimark) *Every C^*-algebra is isomorphic to a C^*-subalgebra of bounded (linear) operators on a (possibly infinite-dimensional) Hilbert space.*

For a proof of theorem 2.4 in the context of quantum groups and noncommutative geometry, see appendix A in [7].

2.3 Quantum $SU(2)$

From here on we take $q \in \mathbf{R} \setminus \{0\}$ (unless otherwise specified) and $k = \mathbf{C}$.

Armed with the technology of section 2.2, we present the quantum group $SU_q(2)$

and certain algebras of functions on it. The constructions and arguments in the section are from Woronowicz's seminal paper [13].

Let A be a C^*-algebra generated by the elements α and γ and satisfying the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1,\ \alpha\alpha^* + q^2\gamma^*\gamma = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*,\ \alpha\gamma = q\gamma\alpha,\ \alpha\gamma^* = q\gamma^*\alpha.$$

We will keep coming back to this set of relations throughout and, for future reference, call it the R-relations (since similar relations also show up in the R-matrix in quantum integrable systems.)

A bare-bone definition of the quantum group $SU_q(2)$ is as the above $*$-algebra A. Of course, one can do very few things with this definition so we make it more analytic and precise.

Let $\mathbf{C}\{\alpha,\gamma,\alpha^*,\gamma^*\}$ be a C^*- algebra generated by the elements α and γ with a unit element and satisfying the R-relations.

By theorem 2.4, $\mathbf{C}\{\alpha,\gamma,\alpha^*,\gamma^*\} \cong O(\mathcal{H})$ where $O(\mathcal{H})$ is a subalgebra of the C^*-algebra of bounded linear operators $B(\mathcal{H})$ on an infinite dimensional Hilbert space $\mathcal{H}$. Therefore we can make the following correspondence (the Gelfand-Naimark-Segal (GNS) representation) $\pi : \mathbf{C}\{\alpha,\gamma,\alpha^*,\gamma^*\} \to O(\mathcal{H})$:

$$\alpha \mapsto \hat{\alpha}, \gamma \mapsto \hat{\gamma}.$$

The elements α^* and γ^* goes to the adjoint operators $\hat{\alpha}^*$ and $\hat{\gamma}^*$ respectively.

If $\hat{\alpha}$ and $\hat{\gamma}$ satisfy the R-relations, we say that π is an admissible representation of $\mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$ on $\mathcal{H}$. By theorem 1.1 in [13], we know that one such admissible representation exists for $\mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$. (In general, to guarantee that there is at least one admissible representation, one has to check whether the Hahn-Banach and the Millman-Krein theorems are satisfied, i.e., whether there are enough pure states on the Hilbert space. For a discussion of these issues, see appendix A of [7].)

For any $a \in \mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$, we define

$$\|a\| = \sup_{\pi} \|\pi(a)\|,$$

where π runs over the set of all admissible representations. One can check that $\|\cdot\| < \infty$ and that $\|\cdot\|$ defines a seminorm. Consider the two-sided ideal N of elements of $a \in \mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$ of vanishing norm and then the quotient algebra

$$\mathcal{A} = \mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}/N.$$

The norm $\|\cdot\|$ induces a norm on $\mathcal{A}$.

Definition 2.5. *A is the completion of $\mathcal{A}$ in this norm.*

Before we state our main theorem of this section, let us present an explicit example (without proof) of the representations of a simpler structure [1]. Consider the C^*-algebra generated by z and satisfying

$$zz^* - zz^* = q(1 - z^*z)(1 - zz^*).$$

Following the construction above, we get a C^* algebra called the *quantum disc.* It has the following two inequivalent admissible representations on some (separable) Hilbert space $\mathcal{H}$:

$$\pi(z) \in \mathbf{C} \subset O(\mathcal{H}), \|z\| = 1, \text{ and } \pi(z) f_n = \sqrt{\frac{qn}{1+qn}} f_{n-1},$$

where $(f_n)_{n=0}^{\infty}$ form an othonormal basis on $\mathcal{H}$.

For the algebra A above, the representations of A are much harder to write down explicitly if we specify a Hilbert space on which they act; typically the representations are written down in terms of q-functions and graded by l an integer or half-integer. (The curious reader is invited to consult the recent preprint of Dąbrowski [4] for a list of the explicit representations of A.)

We now state the main theorem of this section. The algebras A and $\mathcal{A}$ as in above.

Theorem 2.6. (Woronowicz) *Let* $u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$. *Then*

1. *The C^*-algebra $\mathcal{A}$ is generated by the matrix elements of u and is dense in A.*

2. *The matrix u is invertible.*

3. *A is a Hopf *-algebra: there exists C^*-morphisms $\Delta : A \to A \otimes A$ and $S : A \to A$ such that*

$$\Delta u = u \dot{\otimes} u \text{ and } (\mu \circ (id \otimes S) \circ \Delta) u = (\mu \circ (S \otimes id) \circ \Delta) u = u^{-1},$$

where Δ and S are the coproduct and antipode of A respectively.

Before we review the proof of this theorem, let us introduce some terminology.

Definition 2.7. *The pair (A, u) is called the quantum $SU(2)$ group and is denoted as $SU_q(2)$ The matrix u is called the* self representation *of A.*

Proof. (1) follows from the construction of $\mathcal{A}$ above: since A is the completion of $\mathcal{A}$ with respect to the (semi)norm, $\mathcal{A}$ is dense in A.

(2) By definition,

$$u^* = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}^* = \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}.$$

So

$$uu^* = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = u^*u,$$

by using the R-relations. So $u^{-1} = u^*$ and u is invertible.

To prove (3), define the coproduct

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

A direct but tedious computation using the R-relations shows that $\Delta(\alpha)\Delta(\gamma) = q\Delta(\gamma)\Delta(\alpha)$:

$$\Delta(\alpha)\Delta(\gamma) = (\alpha \otimes \alpha - q\gamma^* \otimes \gamma)(\gamma \otimes \alpha + \alpha^* \otimes \gamma)$$

$$= \alpha\gamma \otimes \alpha^2 + \alpha\alpha^* \otimes \alpha\gamma - q\gamma^*\gamma \otimes \gamma\alpha - q\gamma^*\alpha^* \otimes \gamma^2.$$

On the other hand,

$$q\Delta(\gamma)\Delta(\alpha) = q(\gamma \otimes \alpha + \alpha^* \otimes \gamma)(\alpha \otimes \alpha - q\gamma^* \otimes \gamma)$$

$$= \alpha\gamma \otimes \alpha^2 + \alpha\alpha^* \otimes \alpha\gamma - q\gamma^*\gamma \otimes \gamma\alpha - q\gamma^*\alpha^* \otimes \gamma^2,$$

This gives us admissible GNS representations $\widehat{\Delta} : A \to O(\mathcal{H}) \otimes O(\mathcal{H}) \cong O(\mathcal{H} \otimes \mathcal{H})$ from

$$\Delta(\alpha) \mapsto \widehat{\Delta(\alpha)} \text{ and } \Delta(\gamma) \mapsto \widehat{\Delta(\gamma)}$$

and given by

$$\Delta(\widehat{\alpha)\Delta}(\gamma) = q\Delta(\widehat{\gamma)\Delta}(\alpha).$$

This demonstrates that the coproduct Δ induces an admissible representation $\widehat{\Delta}$ in $O(\mathcal{H}) \otimes O(\mathcal{H})$, given an admissible representation $\widehat{(\cdot)}$ on $O(\mathcal{H})$ from generator $(\cdot)$.

Direct computation also checks that $\Delta u = u \dot{\otimes} u$ and $(\mu \circ (\mathrm{id} \otimes S) \circ \Delta)u = u^{-1}$. To check the latter, define the action of the antipode S on the matrix elements of u as

$$S(\alpha) = \alpha^*, S(-q\gamma^*) = \gamma^*, S(\gamma) = -q\gamma \text{ and } S(\alpha^*) = \alpha$$

and use the definition of the inverse of u as above. We check the former computation:

$$u\dot{\otimes}u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\otimes\alpha - q\gamma^*\otimes\gamma & -q\alpha\otimes\gamma^* - q\gamma^*\otimes\alpha^* \\ \gamma\otimes\alpha + \alpha^*\otimes\gamma & -q\gamma\otimes\gamma^* + \alpha^*\otimes\alpha^* \end{pmatrix},$$

while

$$\Delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \Delta(\alpha) & -q\Delta(\gamma^*) \\ \Delta(\gamma) & \Delta(\alpha)^*) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\otimes\alpha - q\gamma^*\otimes\gamma & -q\alpha\otimes\gamma^* - q\gamma^*\otimes\alpha^* \\ \gamma\otimes\alpha + \alpha^*\otimes\gamma & -q\gamma\otimes\gamma^* + \alpha^*\otimes\alpha^* \end{pmatrix}.$$

□

2.4 $SU_q(2)$ and noncommutative geometry

We now discuss the motivations behind the definition of $SU_q(2)$ from a geometrical viewpoint. We will see that the constructions of quantum groups motivate a very radical change in the very way we think of the notion of space in general and differential geometry in particular. This new viewpoint leads to Connes *noncommutative geometry* program ([2] and [3]) and is also mirrored in Manin [9].

Recall that, topologically,

$$SU(2) \approx S^3,$$

where S^3 is the 3-sphere (see [12] for details.) In other words, one views $SU(2)$ as a connected and simply-connected compact manifold that can be readily given a natural differentiable structure. Since S^3 is a compact Hausdorff, we can use the classical Gelfand duality to form a commutative C^*-algebra of continuous functions on S^3, calling it $C(S^3)$. The Gelfand duality is actually an (anti)equivalence of categories [7]:

$$\{\text{compact Hausdorff spaces}\} \simeq \{\text{commutative } C^* \text{ algebras}\}^{op}$$

so homeomorphic compact Hausdorff spaces get carried to isomorphic commutative C^*-algebras by the (contravariant) Gelfand C functor; in particular if X is compact Hausdorff and homeomorphic to S^3, then $C(X) \cong C(S^3)$ as C^*-algebras. In sum, we have the wonderful schematic diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{C \text{ functor}} & C(S^3) \\ \downarrow \approx & & \downarrow \cong \\ SU(2) & \xrightarrow{C \text{ functor}} & C(SU(2)) \end{array}$$

The reader may ask if an analogous situation holds true for $SU_q(2)$. The answer is no, but why this is not the case is fairly interesting.

First of all, there is no obvious "well behaved" topological space that can be identified with $SU_q(2)$: in absence of such an identification, noncommutative geometry takes the radical step of identifying the nonexistent topological space with $SU_q(2)$ itself. This move has precedence in algebraic geometry. As Manin in p. 83 of [9] notes, after Grothendieck, we have learnt that to do geometry on some space, we really do not need the space itself but only a category of sheaves on the space. In case of $SU_q(2)$, we take it to be the space itself characterized by fact that it is a Hopf *-algebra. This is the viewpoint adopted in [9].

However, one can also view the C^*-algebra $\mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$ as the C^*-algebra of functions on some "space", whatever it may be. Upon making this identification in [13], Woronowicz introduces a $\mathbf{C}$-linear map $d : \mathcal{A} \to \Gamma$ where Γ is an $\mathcal{A}$-bimodule. (The C^*-algebra $\mathcal{A}$ is the quotient of $\mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$ by the ideal generated by elements of vanishing norm, as in section 2.3.) The map d acts as a derivation on $\mathcal{A} \times \mathcal{A}$. Morally, $\mathcal{A}$ is viewed as the algebra of "smooth functions" on this (unknown) space and the map d then gives us a first-order differential calculus where $\mathbf{C}\{\alpha, \gamma, \alpha^*, \gamma^*\}$ is the algebra of continous functions.

Connes ([2]) argues that one can, on $\mathcal{A}$, develop many more analogues of differential geometric notions such as an analog of deRham homology theory (called cyclic cohomology), etc. purely on an algebraic level. On the analytic level, many others have followed him in studying *spectral triples* on $SU_q(2)$: a spectral triple on $SU_q(2)$ is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is an infinite dimensional Hilbert space of representations of $\mathcal{A}$, and D an operator on $\mathcal{A}$. Spectral triples encode a lot of geometric information about $SU_q(2)$. (See [4] for a recent review.)

However, information does get lost if one is working purely at the level of C^*-algebras: while the C^*-algebras $C(SU_q(2)) \cong C(SU_{q'}(2))$, in general, the spaces $SU_q(2)$ and $SU_{q'}(2)$ are not isomorphic as Hopf *-algebras. This is sharply different from the commutative case (given by the Gelfand duality) and is the content of a theorem of Wang [11] which we review in the next and final chapter.

Chapter 3

An isomorphism theorem for $SU_q(2)$

3.1 Generalities

In this Chapter we review an isomorphism theorem for $SU_q(2)$ due to Wang: for $q \in \mathbf{R} \setminus \{0\}$, $SU_q(2)$ and $SU_{q'}(2)$ are isomorphic only under very special cases (namely, $q' = q$ or $q' = q^{-1}$).

3.2 Notion of isomorphism

We start with a definition (prompted by the discussion in section 2.4. of chapter 2.)

Definition 3.1. *A* compact quantum group *is a noncommutative and noncocommutative Hopf C^*-algebra.*

The quantum group $SU_q(2)$ is, clearly, an example of a compact quantum group.

Let G and G' be compact quantum groups with coproducts Δ and Δ' respectively. Compact quantum groups form a category with obvious objects and homomorphisms between the underlying C^*-algebras as morphisms such that if $\pi : C(G) \to C(G')$ is such a morphism then the following diagram commutes:

$$\begin{array}{ccc} C(G) \otimes C(G) & \xrightarrow{\pi\otimes\pi} & C(G') \otimes C(G') \\ \uparrow \Delta & & \uparrow \Delta' \\ C(G) & \xrightarrow{\pi} & C(G') \end{array}$$

We say that $G \cong G'$ if π is an isomorphism.

3.3 The isomorphism theorem for $SU_q(2)$

Theorem 3.2. (Wang [11]) *For $q, q' \in \mathbf{R}\backslash\{0\}$, $SU_q(2)$ and $SU_{q'}(2)$ are isomorphic in the sense of section 3.2 if and only if $q' = q$ or $q' = q^{-1}$.*

The proof can be broken up into the proofs of three steps that we prove as lemmas. Our proofs themselves are a reproduction of Wang's, except that for lemma 3.8, we leave out some (technical) details and concentrate instead on giving the reader a flavor of the proof.

Note: The coproduct Δ' on $SU_{q'}(2)$ is given on its generators as

$$\Delta'(\alpha') = \alpha' \otimes \alpha' - q'\gamma'^* \otimes \gamma',$$

$$\Delta'(\gamma') = \gamma' \otimes \alpha' + \alpha'^* \otimes \gamma'.$$

Lemma 3.3. *For $q \in \mathbf{R} \setminus \{0\}$, $SU_q(2) \cong SU_{q^{-1}}(2)$.*

Proof. Let

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ and } u' = \begin{pmatrix} \alpha' & -q^{-1}\gamma'^* \\ \gamma' & \alpha'^* \end{pmatrix}$$

be self representations of $SU_q(2)$ and $SU_{q^{-1}}(2)$ respectively. Define the *-homomorphism

$$\pi : C(SU_q(2)) \to C(SU_{q^{-1}}(2))$$

as $\pi(\alpha) = \alpha'^*$ and $\pi(\gamma) = -q^{-1}\gamma'^*$. (This gives us $\pi(\alpha^*) = \pi(\alpha)^* = \alpha'$ and $\pi(\gamma^*) = \pi(\gamma)^* = -q^{-1}\gamma'$.)

The first observation here is that $\|\pi(\alpha)\| = \|\alpha'^*\| = \|\alpha'\| \neq 0$ and same for γ as well.

We show that π is an isomorphism. Because of the freeness of the underlying $*$-algebras, it suffices to check that π is an isomorphism in the sense of section 3.2 only on the generators.

For α:

$$(\Delta' \circ \pi)\alpha = \Delta'(\pi(\alpha)) = \Delta'(\alpha'^*) = (\alpha' \otimes \alpha' - q^{-1}\gamma'^* \otimes \gamma')^* = \alpha'^* \otimes \alpha'^* - q^{-1}\gamma' \otimes \gamma'^*,$$

whereas

$$(\pi \otimes \pi) \circ \Delta\alpha = (\pi \otimes \pi)(\alpha \otimes \alpha - q\gamma^* \otimes \gamma) = \pi(\alpha) \otimes \pi(\alpha) - q\pi(\gamma^*) \otimes \pi(\gamma)$$

$$= \alpha'^* \otimes \alpha'^* - q(-q^{-1}\gamma'^*)^* \otimes (-q^{-1}\gamma'^*) = \alpha'^* \otimes \alpha'^* - q^{-1}\gamma' \otimes \gamma'^*.$$

So $(\Delta' \circ \pi)\alpha = (\pi \otimes \pi) \circ \Delta\alpha$.

For γ:

$$(\Delta' \circ \pi)\gamma = \Delta'(\pi(\gamma)) = \Delta'(-q^{-1}\gamma'^*) = -q^{-1}\Delta'(\gamma'^*)$$

$$= -q^{-1}(\gamma' \otimes \alpha' + \alpha'^* \otimes \gamma')^* = -q^{-1}\gamma'^* \otimes \alpha'^* - q^{-1}\alpha' \otimes \gamma'^*,$$

whereas

$$(\pi\otimes\pi)\circ\Delta\gamma=(\pi\otimes\pi)(\gamma\otimes\alpha+\alpha^*\otimes\gamma)$$

$$=\pi(\gamma)\otimes\pi(\alpha)+\pi(\alpha^*)\otimes\pi(\gamma)=-q^{-1}\gamma'^*\otimes\alpha'^*+\alpha'\otimes(-q^{-1}\gamma'^*).$$

So $(\Delta'\circ\pi)\gamma=(\pi\otimes\pi)\circ\Delta\gamma$.

This verifes that π is a morphism. To see that π is an isomorphism, define the inverse map

$$\pi^{-1}(\alpha')=\alpha^* \text{ and } \pi^{-1}(\gamma')=-q\gamma^*.$$

We see that

$$\pi\circ\pi^{-1}=\mathrm{id},$$

which verifies that π is an isomorphism. □

To prove the next two lemmas, we will need to use the following facts about *finite dimensional* representations of $SU_q(2)$.

Proposition 3.4. [11] *Let v and w be unitary representations of a compact quantum group. If v and w are equivalent, then $w=CvC^{-1}$ where C is a unitary matrix with complex entries ("unitarily equivalent").*

Proposition 3.5. [11] *Let $\pi: C(G)\to C(G')$ be a morphism of compact quantum groups and u a matrix representation of G. Then $\pi(u)$ is a representation of G'.*

Proposition 3.6. (Classfication of representations of $SU_q(2)$ [13]) *$SU_q(2)$ has only one 2-dimensional irreducible representation upto equivalence.*

Lemma 3.7. *For $q \in [-1,1] \setminus \{0\}$, (a) $SU_q(2) \cong SU(2) \iff q = 1$ and (b) $SU_q(2) \cong SU_{-1}(2) \iff q = -1$.*

Proof. Part (a): For $q = 1$, $C(SU_q(2)$ is commutative while for $q \neq 1$, $C(SU_q(2))$ is noncommutative. Therefore $SU_q(2) \cong SU(2) \iff q = 1$.

Part (b): Assume the contrary, i.e., assume that there exists an isomorphism $\pi : C(SU_{-1}(2)) \to C(SU_q(2))$ for some $q \in (-1,1) \setminus \{0\}$. Then $\pi(u)$ is a 2-dimensional irreducible representation of $SU_q(2)$, where u is the self representation of $SU_{-1}(2)$ (by propositions 3.5 and 3.6.) Now by proposition 3.6 again, $SU_q(2)$ has only one 2-dimensional irreducible representation u', so by proposition 3.4, $\pi(u)$ is unitarily equivalent to u'.

Passing on to the conjugate representation of u, since u is unitary, $\bar{u}$ is unitary so $\pi(\bar{u})$ is unitary. Therefore, $\bar{u}'$ must be unitary (since π is an isomorphism). This happens if and only if $q^2 = 1 \Rightarrow q = \pm 1$, so we have a contradiction. □

Lemma 3.8. *For $q \in [-1,1] \setminus \{0\}$, $SU_q(2) \cong SU_{q'}(2) \iff q = q'$.*

Proof. (outline) Let

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ and } u' = \begin{pmatrix} \alpha' & -q'\gamma'^* \\ \gamma' & \alpha'^* \end{pmatrix}$$

be self representations of $SU_q(2)$ and $SU_{q'}(2)$ respectively. By the propositions 3.4, 3.5 and 3.6,

$$u' = C\pi(u)C^{-1},$$

where C is a complex unitary 2×2 matrix. So $u' = C\pi(u)C^*$. This gives us a system of 8 equations in the variables c_{ij} and q and q' (4 for c_{ij} and 4 for $\bar{c_{ij}}$) by comparing the coefficients of $\alpha', \alpha'^*, \gamma'$ and γ'^* on both sides. Solving this system gives us $q' = q$. □

Proof. (*of theorem 3.2.*) Lemmas 3.3, 3.7 and 3.8. □

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